PHY 4604 Spring 2013 — Exam 2

1(a) Matrix Ω_{mn} corresponds to outer product $\hat{\Omega} = \sum_{m,n} \Omega_{mn} |\chi_m\rangle\langle\chi_n|$.

In the present question

$$\hat{G} = \gamma\left(2|2\rangle\langle 2| - |1\rangle\langle 1|\right)$$

$$\hat{H} = \epsilon\left[3\left(|1\rangle\langle 1| + |2\rangle\langle 2|\right) + 2i\left(|2\rangle\langle 1| - |1\rangle\langle 2|\right)\right]$$

(b) Energy eigenvalues E satisfy the characteristic equation

$$0 = \det(\hat{H} - E\hat{I}) = \begin{vmatrix} 3\epsilon - E & -2i\epsilon \\ 2i\epsilon & 3\epsilon - E \end{vmatrix} = (3\epsilon - E)^2 - |2i\epsilon|^2$$

$$E = 3\epsilon \pm 2\epsilon$$

Choose $E_1 = \epsilon$, $E_2 = 5\epsilon$.

$$E = E_1: \begin{pmatrix} 2\epsilon & -2i\epsilon \\ 2i\epsilon & 2\epsilon \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \text{eigenvector} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \frac{e^{i\Theta_1}}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}$$

$$E = E_2: \begin{pmatrix} -2\epsilon & -2i\epsilon \\ 2i\epsilon & -2\epsilon \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \text{eigenvector} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \frac{e^{i\Theta_2}}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}$$

Choose $|E_1\rangle = \frac{1}{\sqrt{2}}(|1\rangle - i|2\rangle)$ and $|E_2\rangle = \frac{1}{\sqrt{2}}(|1\rangle + i|2\rangle)$

(c) From (b), $|1\rangle = \frac{1}{\sqrt{2}}(|E_1\rangle + |E_2\rangle)$ and $|2\rangle = \frac{i}{\sqrt{2}}(|E_1\rangle - |E_2\rangle)$

Since $|\psi(0)\rangle = e^{i\pi/3}|2\rangle = e^{i\pi/3}\frac{i}{\sqrt{2}}(|E_1\rangle - |E_2\rangle)$

$$\Rightarrow |\psi(t)\rangle = e^{i\pi/3}\frac{i}{\sqrt{2}}\left(e^{-iE_1t/\hbar}|E_1\rangle - e^{-iE_2t/\hbar}|E_2\rangle\right)$$

$$= e^{i\pi/3}\frac{i}{\sqrt{2}}\left[e^{-i\epsilon t/\hbar}\frac{1}{\sqrt{2}}(|1\rangle - i|2\rangle) - e^{-5i\epsilon t/\hbar}\frac{1}{\sqrt{2}}(|1\rangle + i|2\rangle)\right]$$

$$= e^{i\left(\frac{\pi}{3} - \frac{3\epsilon t}{\hbar}\right)}\left[\frac{i}{2}\left(e^{2i\epsilon t/\hbar} - e^{-2i\epsilon t/\hbar}\right)|1\rangle + \frac{1}{2}\left(e^{2i\epsilon t/\hbar} + e^{-2i\epsilon t/\hbar}\right)|2\rangle\right]$$

$$= \exp\left[i\left(\frac{\pi}{3} - \frac{3\epsilon t}{\hbar}\right)\right]\left[\cos\left(\frac{2\epsilon t}{\hbar}\right)|2\rangle - \sin\left(\frac{2\epsilon t}{\hbar}\right)|1\rangle\right]$$

(d) $\hat{G}$ is diagonal in basis $\{|1\rangle, |2\rangle\}$ so it clearly has eigenvalues $G_1 = -\gamma$ and $G_2 = 2\gamma$ with eigenvectors $|G_1\rangle = |1\rangle$ and $|G_2\rangle = |2\rangle$.

A measurement of G at time t can yield one of the eigenvectors

$G = G_1 = -\gamma$ with probability $P(G_1) = |\langle G_1|\psi(t)\rangle|^2 = \sin^2\frac{2\epsilon t}{\hbar}$

or $G = G_2 = 2\gamma$ " " $P(G_2) = |\langle G_2|\psi(t)\rangle|^2 = \cos^2\frac{2\epsilon t}{\hbar}$

$$\text{Expectation value} \quad \langle G \rangle = \sum_j G_j P(G_j) = 2\gamma \cos^2 \frac{2\epsilon t}{\hbar} - \gamma \sin^2 \frac{2\epsilon t}{\hbar}$$
$$= \gamma \left[3\cos^2\left(\frac{2\epsilon t}{\hbar}\right) - 1 \right] \quad \text{using } \cos^2 + \sin^2 = 1$$

This is minimized when
$$\cos \frac{2\epsilon t}{\hbar} = 0 \quad \text{or} \quad t = \frac{(2n+1)\pi\hbar}{4\epsilon} \quad n = 0, 1, 2, \ldots$$

2(a) For any $V(x)$, bound states exist only at energies E in the range
$$\inf V(x) < E < \min\left[V(\infty), V(-\infty)\right]$$
which in this case means $E < 0$.

(b) Since $E < V(x)$ for both $x > 0$ and $x < 0$, the wave function is of the form
$$\psi(x) = \begin{cases} A_1 e^{K_1 x} & \text{for } x \leq 0 \text{ with } K_1 = \sqrt[+]{-2mE}/\hbar \\ B_2 e^{-K_2 x} & \text{for } x \geq 0 \text{ with } K_2 = \sqrt[+]{2m(S-E)}/\hbar \end{cases}$$

(c) Boundary conditions at $x = 0$:
$$\Delta\psi = 0 \quad \Rightarrow \quad B_2 - A_1 = 0 \quad ①$$
$$\Delta\psi' = \frac{2m}{\hbar^2}\alpha\psi \quad \Rightarrow \quad -K_2 B_2 - K_1 A_1 = -\frac{2m}{\hbar^2}\alpha A_1 \quad ②$$
$$①, ② \Rightarrow \quad K_1 + K_2 = \frac{2m}{\hbar^2}\alpha$$
$$\times\sqrt{\frac{\hbar^2}{2m}} \Rightarrow \quad \sqrt{-E} + \sqrt{S-E} = \sqrt{\frac{2m\alpha^2}{\hbar^2}} \quad ③$$

(d) From (a), the bound-state energy must satisfy $E < 0$
$$\Rightarrow \quad \sqrt{-E} + \sqrt{S-E} > \sqrt{S} \quad ④$$
$$③, ④ \Rightarrow \quad S < \frac{2m\alpha^2}{\hbar^2} = S_{max}$$

So no bound state for $S \geq S_{max}$

(e) Squaring both sides of ③,
$$-E + S - E + 2\sqrt{-E(S-E)} = S_{max}$$
$$2\sqrt{-E(S-E)} = S_{max} - S + 2E$$
$$\text{Square:} \quad -\cancel{4SE} + \cancel{4E^2} = (S_{max} - S)^2 + 4S_{max}E - \cancel{4SE} + \cancel{4E^2}$$
$$E = -\frac{(S_{max} - S)^2}{4S_{max}}$$
$$= -\frac{\hbar^2}{8m\alpha^2}\left(\frac{2m\alpha^2}{\hbar^2} - S\right)^2$$
$$= -\frac{m\alpha^2}{2\hbar^2}\left(1 - \frac{\hbar^2 S}{2m\alpha^2}\right)^2$$