PHY 4604 Spring 2013 – Final Exam

1. The angular momentum depends only on the angular wave function

$$Y(\theta,\phi) = A\,(1 + \sqrt{24}\,\sin\theta\,\sin\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} c_l^m\, Y_l^m(\theta,\phi)$$

Using the formula sheet

$$Y(\theta,\phi) = A\left[(4\pi)^{1/2} Y_0^0 + \sqrt{24}\,\frac{i}{2}\cdot\left(\frac{8\pi}{3}\right)^{1/2} Y_1^1 + \sqrt{24}\,\frac{i}{2}\left(\frac{8\pi}{3}\right)^{1/2} Y_1^{-1}\right]$$

$$= \sqrt{4\pi}\,A\,(Y_0^0 + 2i\,Y_1^1 + 2i\,Y_1^{-1})$$

$$= \frac{1}{3}(Y_0^0 + 2i\,Y_1^1 + 2i\,Y_1^{-1}) \quad \text{up to an overall phase.}$$

Since Y_l^m has $L_z = m\hbar$, possible outcomes of the measurement are

$L_z = 0$ with probability $\left|\frac{1}{3}\right|^2 = \frac{1}{9}$

$L_z = \hbar$ " " $\left|\frac{2i}{3}\right|^2 = \frac{4}{9}$

$L_z = -\hbar$ " " $\left|\frac{2i}{3}\right|^2 = \frac{4}{9}$

2(a) Expectation value of spin component $j \in \{x, y, z\}$ is

$$\langle S_j\rangle = \chi^\dagger(t)\,\hat{S}_j\,\chi(t) = \frac{\hbar}{2}\chi^\dagger(t)\,\sigma_j\,\chi(t)$$

$$\langle S_x\rangle = \frac{\hbar}{2}\frac{1}{\sqrt{2}}(c+s \quad c-s)\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}c+s\\ c-s\end{pmatrix} \qquad \begin{matrix} c = \cos\omega t \\ s = \sin\omega t\end{matrix}$$

$$= \frac{\hbar}{2}(c+s)(c-s) = \frac{\hbar}{2}(c^2 - s^2)$$

$$= \frac{\hbar}{2}\cos(2\omega t)$$

$$\langle S_y\rangle = \frac{\hbar}{2}\frac{1}{\sqrt{2}}(c+s \quad c-s)\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}c+s\\ c-s\end{pmatrix}$$

$$= 0$$

$$\langle S_z\rangle = \frac{\hbar}{2}\frac{1}{\sqrt{2}}(c+s \quad c-s)\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}c+s\\ c-s\end{pmatrix}$$

$$= \frac{\hbar}{4}\left[(c+s)^2 - (c-s)^2\right] = \hbar cs$$

$$= \frac{\hbar}{2}\sin(2\omega t)$$

(b) $\langle\vec{S}\rangle$ rotates in the xz plane around the y axis.

(c) From (a), the angular frequency of precession is 2ω.

3(a) The radial probability density in a stationary state of a spherical pot^{l} is

$$P(r) = r^2 |R(r)|^2 = |A|^2 r^{2l+2} \exp\left[-\frac{2r}{(l+1)a_0}\right]$$

The most probable radius r_m is the radius that maximizes P:

$$0 = \left.\frac{dP}{dr}\right|_{r=r_m}$$

$$= |A|^2 \left\{(2l+2) r_m^{2l+1} - \frac{2}{(l+1)a_0} r_m^{2l+2}\right\} \exp\left[-\frac{2r_m}{(l+1)a_0}\right]$$

The contents of the curly braces must vanish:

$$\Rightarrow \quad r_m = (l+1)^2 a_0$$

(b)

$$\langle r \rangle = \int_0^\infty r P(r)\, dr = |A|^2 \int_0^\infty r^{2l+3} \exp\left[-\frac{2r}{(l+1)a_0}\right] dr$$

Changing variables from r to

$$\rho = \frac{2r}{(l+1)a_0}$$

$$\Rightarrow \quad \langle r \rangle = |A|^2 \left[\tfrac{1}{2}(l+1)a_0\right]^{2l+4} \underbrace{\int_0^\infty \rho^{2l+3} e^{-\rho}\, d\rho}_{(2l+3)!} \qquad ①$$

Also

$$1 = \int_0^\infty P(r)\, dr = |A|^2 \int_0^\infty r^{2l+2} \exp\left[-\frac{2r}{(l+1)a_0}\right] dr$$

$$= |A|^2 \left[\tfrac{1}{2}(l+1)a_0\right]^{2l+3} \underbrace{\int_0^\infty \rho^{2l+2} e^{-\rho}\, d\rho}_{(2l+2)!} \qquad ②$$

$$① \div ② \Rightarrow \quad \langle r \rangle = (l + \tfrac{3}{2})(l+1) a_0 \qquad \text{(greater than } r_m\text{)}$$

(c)

$$\langle r^2 \rangle = \int_0^\infty r^2 P(r)\, dr = |A|^2 \int_0^\infty r^{2l+4} \exp\left[-\frac{2r}{(l+1)a_0}\right] dr$$

$$= |A|^2 \left[\tfrac{1}{2}(l+1)a_0\right]^{2l+5} \underbrace{\int_0^\infty \rho^{2l+4} e^{-\rho}\, d\rho}_{(2l+4)!} \qquad ③$$

$$③ \div ② \Rightarrow \quad \langle r^2 \rangle = (l+2)(l+\tfrac{3}{2})(l+1)^2 a_0^2$$

Variance

$$\sigma_r^2 = \langle r^2 \rangle - \langle r \rangle^2$$

$$= \left[(l+2)(l+\tfrac{3}{2}) - (l+\tfrac{3}{2})^2\right](l+1)^2 a_0^2$$

$$= \tfrac{1}{2}(l+\tfrac{3}{2})(l+1)^2 a_0^2$$

Uncertainty

$$\sigma_r = \sqrt{\frac{2l+3}{4}}\,(l+1) a_0$$