PHY 4604 Spring 2013 – Homework 1

Due at the start of class on Friday, January 18. No credit will be available for homework submitted after the start of class on Wednesday, January 23.

Answer all four questions. Please write neatly and include your name on the front page of your answers. You must also clearly identify all your collaborators on this assignment. To gain maximum credit you should explain your reasoning and show all working.

This assignment is primarily designed to provide practice with standard mathematical techniques encountered in wave mechanics. You may find useful the following integrals:

$$\int \sin^2 x\,dx = \frac{1}{2}(x - \sin x \cos x), \qquad \int_0^\infty x^{2n} \exp(-x^2/a^2)\,dx = \sqrt{\pi}\,\frac{(2n)!}{n!}\left(\frac{a}{2}\right)^{2n+1}.$$

In the second equation, a is real, n is a non-negative integer, and $n! = n \cdot (n-1)!$ with $0! = 1$.

1. A point-like particle of mass m moving in one dimension is confined between hard walls at $x = 0$ and $x = L$. The particle is described by the wave function

$$\Psi(x,t) = \begin{cases} A \sin(2\pi x/L) \exp(-iEt/\hbar) & \text{for } 0 \le x \le L, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

   Here L is a real length, while the constants A and E may be real, imaginary, or complex.

   (a) Find a choice of A that normalizes the wave function. What condition must be satisfied to ensure that A is truly a constant, i.e., it is independent of time?

   (b) Assuming that the condition for A to be constant is met, calculate the probability that a measurement of the particle's position x at time t yields a result $x(t) < L/4$.

2. Suppose that within some region $a < x < b$, a particle of mass m experiences a potential $V(x,t) = 0$ and is described by a linear superposition of two plane-wave functions:

$$\Psi(x,t) = c_1 e^{i(k_1 x - \omega_1 t + \theta_1)} + c_2 e^{i(k_2 x - \omega_2 t + \theta_2)},$$

   where (for $\alpha = 1$, 2), c_α, k_α, and θ_α are real constants, and $\omega_\alpha = \hbar k_\alpha^2/2m$. This wave function generalizes an example of linear superpositions that was considered in class. If $k_\alpha > 0$ ($k_\alpha < 0$), then component α of the wave function represents a right-moving (left-moving) wave. [Note that if this form of $\Psi(x,t)$ were to extend all the way to $|x| = \infty$, the wave function would not be normalizable. We will assume that $\Psi(x,t) \to 0$ for $x \ll a$ and for $x \gg b$.]

   (a) Calculate the probability density $\rho(x,t)$ and the probability current $j(x,t)$ within the region $a < x < b$. Cast your answers in forms that make clear that both quantities are real.

   (b) Verify by substitution of your results from (a) that they satisfy the continuity equation

$$\frac{\partial \rho}{\partial t} = -\frac{\partial j}{\partial x} + \frac{2\,\mathrm{Im}V}{\hbar}\rho.$$

(c) Under what condition(s) will $\rho(x,t)$ and $j(x,t)$ be independent of position?

(d) Under what condition(s) will $\rho(x,t)$ and $j(x,t)$ be independent of time?

(e) Under what condition(s) will $j(x,t) = 0$ throughout the region $a < x < b$?

In answering parts (c)–(e), make sure that you avoid the trivial case where $c_1 = 0$ and $c_2 = 0$, because this just means $\Psi(x,t) = 0$, i.e., there is no particle!

3. Consider the wave function

$$\Psi(x,t) = A\, \exp[-(x^2/2a^2 + i\omega t)], \tag{2}$$

where a is a real length scale and ω is a real angular frequency.

(a) Find the positive, real constant A that normalizes the wave function.

(b) Calculate $\langle x\rangle$, $\langle x^2\rangle$, and σ_x.

(c) Calculate $\langle p\rangle$, $\langle p^2\rangle$, and σ_p.

(d) Verify that σ_x and σ_p satisfy the uncertainty principle.

4. By (i) differentiating the momentum expectation value

$$\langle p\rangle = \int_{-\infty}^{\infty} \Psi^*(x,t)\; (-i\hbar\, \partial/\partial x)\; \Psi(x,t)\, dx$$

with respect to time, (ii) replacing $\partial\Psi/\partial t$ and $\partial\Psi^*/\partial t$ using the Schrödinger wave equation, and (iii) integrating by parts to cancel most of the terms, prove that

$$d\langle p\rangle/dt = \langle F(x,t)\rangle, \tag{3}$$

where $F(x,t) = -\partial V(x,t)/\partial x$ is the classical force corresponding to a real, conservative potential $V(x,t)$.

Additional information (no work required on the part of the student): Equation (3) is consistent with *Ehrenfest's theorem*, which provides a general expression for the time evolution of the expectation value of any physical property. Ehrenfest's theorem is commonly stated in words along the lines (e.g., see Griffiths Problem 1.7) "expectation values obey classical laws," implying that a quantum-mechanical equation of motion can be obtained from the corresponding classical one via the replacements $x \to \langle x\rangle$ and $p \to \langle p\rangle$. This recipe certainly works for the equation $d\langle x\rangle/dt = \langle p\rangle/m$ derived in class.

However, by expanding $F(x,t)$ in a Taylor series about $x = \langle x\rangle$, the expected position at time t, one can transform Eq. (3) to

$$d\langle p\rangle/dt = F(\langle x\rangle, t) + \tfrac{1}{2}\,\sigma_x^2\, F''(\langle x\rangle, t) + \ldots, \tag{4}$$

where σ_x is the position uncertainty at time t and $F''(x,t) = \partial^2 F(x,t)/\partial x^2$. The term $F(\langle x\rangle, t)$ is the result of applying $x \to \langle x\rangle$ and $p \to \langle p\rangle$ to Newton's second law. The other terms on the right-hand side of Eq. (4) arise because the weighted average of the force over the position probability distribution is, in general, not the same as the force evaluated at the average position. The presence of these additional terms disproves any general notion that "expectation values obey classical laws." What is true is that Newton's second law is recovered in the limit $\sigma_x \to 0$ where the wave function has negligible spatial extent.