PHY 4604 Spring 2013 – Homework 2

Due at the start of class on Friday, February 1. No credit will be available for homework submitted after the start of class on Wednesday, February 6.

Answer both questions. Please write neatly and include your name on the front page of your answers. You must also clearly identify all your collaborators on this assignment. To gain maximum credit you should explain your reasoning and show all working.

Your may find useful the following mathematical results:

$$\sin^2 x + \cos^2 x = 1, \quad 2 \sin x \cos y = \sin(x-y) + \sin(x+y)$$

$$2 \sin x \sin y = \cos(x-y) + \cos(x+y), \quad 2 \cos x \cos y = \cos(x-y) - \cos(x+y)$$

$$\int x \sin x \, dx = \sin x - x \cos x, \quad \int x \cos x \, dx = x \sin x + \cos x, \quad \int \sin^2 x \, dx = \frac{x}{2} - \frac{1}{4} \sin 2x$$

$$\int x^2 \cos x \, dx = (x^2-2) \sin x + 2x \cos x, \quad \int x^2 \sin^2 x \, dx = \frac{x^3}{6} - \frac{2x^2-1}{8} \sin 2x - \frac{x}{4} \cos 2x$$

1. An infinite square well confines a particle of mass m to the region $-a/2 < x < a/2$. (This well is shifted compared to the one considered in class.) The particle's spatial wave functions can be written $\psi_n(x) = \sqrt{2/a} \cos(n\pi x/a)$ for $n = 1, 3, 5, \ldots$, and $\psi_n(x) = \sqrt{2/a} \sin(n\pi x/a)$ for $n = 2, 4, 6, \ldots$. Therefore, $\psi_n(-x) = (-1)^{n-1} \psi_n(x)$, a relationship that holds [with $(-1)^{n-1}$ replaced by $(-1)^n$ in cases where the ground state is labeled $n = 0$ rather than $n = 1$] for any potential satisfying $V(-x) = V(x)$. Throughout the parts below, take advantage of symmetries and other simplifications to minimize the number of integrals that you must perform by brute force.

   (a) Calculate $\langle x \rangle$, $\langle x^2 \rangle$, and σ_x in the n^{th} stationary state.

   (b) Calculate $\langle p \rangle$, $\langle p^2 \rangle$, and σ_p in the n^{th} stationary state. Check that the uncertainty principle is obeyed.

   (c) Suppose that the system's initial state is $\Psi(x, t = 0) = [\psi_1(x) + 2\psi_2(x)] / \sqrt{5}$.

      i. Find the probability that a position measurement performed at $t = 0$ yields a result $x > 0$.

      ii. Find the wave function $\Psi(x, t)$ at time $t > 0$.

      iii. Calculate $\langle x \rangle$ at time $t \geq 0$. Show that this quantity oscillates, and identify the amplitude and the angular frequency of the oscillation.

      iv. Calculate $\langle p \rangle$ at time $t \geq 0$. You can reduce your work by applying the result derived in class: $\langle p \rangle = m \, d\langle x \rangle / dt$.

   (d) Suppose instead that the system's initial state is $\Psi(x, 0) = [\psi_1(x) + 2\psi_3(x)] / \sqrt{5}$. Argue, without performing a detailed calculation, that in this case $\langle x \rangle$ does not change with time.

(e) Suppose instead that the system's initial state is

$$\Psi(x,0) = \begin{cases} A\,(a/2 - |x|) & \text{for } |x| < a/2, \\ 0 & \text{otherwise.} \end{cases}$$

Find the normalization constant A and the wave function $\Psi(x,t)$ at time $t > 0$.

2. The "shooting method" for finding solutions of the 1D time-independent Schrödinger wave equation entails choosing a trial solution $\psi(x)$ for large, negative x and then integrating $d^2\psi(x)/dx^2 = (2m/\hbar^2)[V(x) - E]\psi(x)$ to obtain the full solution. A physically acceptable solution must satisfy three conditions: (i) it must be normalizable, so $\psi(x)$ must vanish sufficiently fast as $x \to \pm\infty$; (ii) $\psi(x)$ must be continuous everywhere; and (iii) $d\psi/dx$ must be continuous everywhere except at the location $x = x_j$ of any infinite jump in $V(x)$, where $\psi(x_j) = 0$ is the only solution. We can use the shooting method to find a qualitative picture of solutions $\psi(x)$ in many cases where obtaining a full analytical solution would be messy or even impossible.

   Consider a particle of mass m confined by the potential

$$V(x) = \begin{cases} 0 & \text{for } -a/2 < x < 0, \\ -v & \text{for } 0 < x < a/2, \\ \infty & \text{for } |x| > a/2. \end{cases}$$

   Answer the questions below using qualitative arguments, not detailed calculations. The wave functions ψ_a, ψ_b, and ψ_c defined in parts (a)–(c) should all be sketched (not plotted) and clearly labeled **on the same graph**, with a brief explanation elsewhere of how you reached your answers to (b) and (c). To set the horizontal and vertical scales of the graph, label 0 and $\pm a/2$ and label $\pm\sqrt{2/a}$ on the ψ axis.

   (a) Sketch $\psi_a(x)$, the (normalized) ground-state ($n = 1$) wave function for the case $v = 0$ where the potential reduces to the infinite square well.

   (b) Now consider the case $v \simeq \hbar^2\pi^2/(10ma^2)$. (This approximate value is provided to give you an idea of the size of v relative to the energy of the state ψ_a.) Let $\psi_b(x)$ be the solution of the time-independent Schrödinger wave equation for this $v > 0$ that coincides exactly with the $v = 0$ solution $\psi_a(x)$ throughout the region $x < 0$. Sketch the qualitative form of $\psi_b(x)$ over the entire range $-a/2 < x < a/2$.

   It is likely that $\psi_b(x)$ is not properly normalized, but this can easily be rectified by multiplying the wave function by the appropriate constant. Give a second reason reason why $\psi_b(x)$ is not a physically valid spatial wave function for this potential.

   (c) Sketch $\psi_c(c)$, the (normalized) ground-state wave function for the value of v considered in (b). Make clear the qualitative differences between $\psi_c(x)$ and the other two wave functions that you have sketched.

   (d) Will the probability density at $x = -a/4$ be greater or smaller for $v > 0$ than for $v = 0$, or will it be the same in the two cases? What about the probability density at $x = a/4$? In each case, briefly explain your reasoning.