PHY 4604 Spring 2013 – Homework 4

Due by 4:00 p.m. on Tuesday, March 12. Please deliver your homework to NPB 2162, or push it under the door if the room is unoccupied. No credit will be available for homework submitted after the start of class on Friday, March 15.

Answer all three questions. Please write neatly and include your name on the front page of your answers. You must also clearly identify all your collaborators on this assignment. To gain maximum credit you should explain your reasoning and show all working.

1. A particle of mass m moves in the one-dimensional potential

$$V(x) = -V_0 a \delta(x),$$

where $V_0 > 0$ is an energy and $a > 0$ is a length scale. The bound state wave function for this problem was discussed in class; see also Eq. (2.129) of Griffiths (where $\alpha \equiv V_0 a$).

   (a) Find $\langle V \rangle$, the expectation value of the potential in the bound state.

   (b) Find $\langle T \rangle$, the expectation value of the kinetic energy in the bound state, where $\hat{T} = -(\hbar^2/2m)\partial^2/\partial x^2$. Verify that $\langle T \rangle + \langle V \rangle = E$, the energy of the bound state. Hint: In order to obtain the correct value for $\langle T \rangle$, you must take into account when evaluating $\partial^2\Psi/\partial x^2$ the fact that the slope of the wave function undergoes a jump at $x = 0$.

   (c) Prove, through integration by parts, that one can also obtain the expectation value of the kinetic energy in any state $\Psi(x,t)$ as

$$\langle T \rangle = \frac{\hbar^2}{2m} \int_{-\infty}^{\infty} \left| \frac{\partial \Psi(x,t)}{\partial x} \right|^2 dx.$$

   Apply this result to the bound state of the delta-function potential and verify that it gives the same result as you obtained in part (b).

2. A particle of mass m moves in an infinite square well that contains a delta-function barrier in its middle. The potential is

$$V(x) = \begin{cases} \dfrac{v\hbar^2}{ma}\delta(x) & \text{for } |x| < a/2, \\ \infty & \text{for } |x| \geq a/2, \end{cases}$$

where $v > 0$ is a pure number that characterizes the strength of the barrier. Since $V(x) = V(-x)$, any bound state must satisfy either $\psi(x) = \psi(-x)$ (even parity) or $\psi(x) = -\psi(-x)$ (odd parity).

   (a) Write down the general form of an **even-parity** bound-state wave function of energy E in terms of unknown amplitudes and quantities specified in the question.

   (b) Apply boundary conditions at $x = 0$ and $x = \pm a/2$ to (i) express the **even-parity** bound state in terms of a single unknown amplitude; and (ii) find a condition that determines the allowed bound-state energies.

(c) Use a method of your choice (e.g., numerical substitution of trial values into the equation that you found in the previous part) to determine the energies of the first two **even-parity** bound states for the particular case $v = 2$. Express each energy as a multiple of $\hbar^2/(2ma^2)$, accurate to three significant figures.

(d) Repeat (a) for an **odd-parity** bound state.

(e) Repeat (b) for an **odd-parity** bound state.

(f) Repeat (c) for an **odd-parity** bound state.

3. A particle of mass m moves in the one-dimensional potential

$$V(x) = V_0\Theta(x) + V_1 a\delta(x),$$

where V_0 and V_1 are positive energy scales, $a > 0$ is a length scale, and $\Theta(x)$ is the Heaviside or step function:

$$\Theta(x) = \begin{cases} 0 & \text{for } x < 0, \\ 1 & \text{for } x > 0. \end{cases}$$

This question focuses on stationary states with an energy $E > V_0$.

(a) Write down the general form of the wave function in the regions $x < 0$ and $x > 0$.

(b) Apply the appropriate boundary conditions to construct the wave function $\psi_r(x)$ describing a rightward moving particle incident from the far left, plus any reflected and transmitted products.

(c) Find the transmission probability $T(E)$.