PHY 4604 Spring 2013 – Homework 5

Due by 4:00 p.m. on Tuesday, March 26. Please deliver your homework to NPB 2162, or push it under the door if the room is unoccupied. No credit will be available for homework submitted after the start of class on Friday, March 29.

Answer both questions. Please write neatly and include your name on the front page of your answers. You must also clearly identify all your collaborators on this assignment. To gain maximum credit you should explain your reasoning and show all working.

1. *Half potentials.* Any potential $V_s(x)$ that is mirror symmetric about $x = 0$, i.e., $V_s(-x) = V_s(x)$ for all x has N_s bound states labeled $n = 0, 1, 2, \ldots, N_s - 1$ with energies $E_{s,n}$ that satisfy $E_{s,n+1} > E_{s,n}$ and with normalized wave functions $\psi_{s,n}(x)$ that satisfy $\psi_{s,n}(-x) = (-1)^n \psi_{s,n}(x)$.

   Then the "half potential" $V_h(x)$ formed from $V_s(x)$ by inserting a hard wall that prevents the particle entering the region $x < 0$, i.e.,

   $$V_h(x) = \begin{cases} V_s(x) & \text{for } x > 0, \\ \infty & \text{for } x \le 0, \end{cases}$$

   has $N_h = \lfloor N_s/2 \rfloor$ bound states labeled $n = 0, 1, 2, \ldots, N_h - 1$ with energies $E_{h,n} = E_{s,2n+1}$ and normalized wave functions

   $$\psi_{h,n}(x) = \begin{cases} \sqrt{2}\, \psi_{s,2n+1}(x) & \text{for } x \ge 0, \\ 0 & \text{for } x \le 0. \end{cases}$$

   Here $\lfloor x \rfloor$ is the greatest integer less than or equal to x.

   The preceding statements can be applied to the potential

   $$V(x) = \begin{cases} \infty & \text{for } x \le 0, \\ -V_0 & \text{for } 0 < x < a, \\ 0 & \text{for } x \ge a, \end{cases}$$

   where $V_0 > 0$.

   (a) Write down two equations whose simultaneous solution determine the bound-state energies E_n of this potential. You may quote the relevant equations based on the standard treatment of a potential related to $V(x)$.

   (b) Show that there is no bound state of the potential $V(x)$ unless V_0 exceeds a minimum value. What is this value?

2. *Quantum mechanics in a two-dimensional Hilbert space.* A quantum mechanical system is described by a two-dimensional vector space. The Hamiltonian operator for this system is $\hat{H} = i\Delta\big(|\omega_1\rangle\langle\omega_2| - |\omega_2\rangle\langle\omega_1|\big)$, where $|\omega_1\rangle$ and $|\omega_2\rangle$ are the eigenstates of the observable operator $\hat{\Omega}$ corresponding to eigenvalues $\omega_1 \neq \omega_2$.

   (a) What condition(s) must the scalars ω_1, ω_2, and Δ satisfy?

   (b) Provide the matrix representations of $\hat{H}$ and $\hat{\Omega}$ in the basis $\{|\omega_1\rangle, |\omega_2\rangle\}$.

   (c) Find the eigenvalues E_1 and E_2 ($E_1 < E_2$) of $\hat{H}$ and their corresponding normalized eigenstates $|E_1\rangle$ and $|E_2\rangle$. Express the eigenstates as kets, not as column vectors.

   (d) Express $|\omega_1\rangle$ and $|\omega_2\rangle$ as linear combinations of $|E_1\rangle$ and $|E_2\rangle$.

   (e) Use your answers from (d) to express $\hat{H}$ and $\hat{\Omega}$ in the basis $\{|E_1\rangle, |E_2\rangle\}$. Give your answers both in outer-product form and in matrix form.

   (f) A system that at time zero is in a state $|E_n\rangle$ will at time t be in the state $\exp(-iE_nt/\hbar)|E_n\rangle$. Use this fact to construct the *time-evolution operator* $\hat{U}(\tau)$ that converts the state vector at time t to the state vector at time $t+\tau$, i.e., $\hat{U}(\tau)|\Psi(t)\rangle = |\Psi(t+\tau)\rangle$ for any $|\Psi(t)\rangle$. Give your answer both in outer-product form and as a matrix in the basis $\{|E_1\rangle, |E_2\rangle\}$.

   For the remainder of the question, assume that the system is described at time $t = 0$ by a state vector $|\Psi(0)\rangle = |\omega_1\rangle$.

   (g) Find $|\Psi(t)\rangle$ for some later time $t > 0$. Express your result as a ket in the basis $\{|E_1\rangle, |E_2\rangle\}$.

   (h) Express $|\Psi(t)\rangle$ as a ket in the basis $\{|\omega_1\rangle, |\omega_2\rangle\}$.

   (i) What are the possible outcomes of a measurement of the energy performed at time t, and what is the probability of each outcome?

   (j) What are the possible outcomes of a measurement of Ω performed at time t, and what is the probability of each outcome?

   (k) Find the expectation values of H and Ω at time t.

   This question exemplifies the process changing the basis used to describe vectors in a Hilbert space. Sections A.4 and A.5 of Griffiths provide a more general discussion of basis changes using unitary (inner-product-conserving) transformations.

   It should be clear from the spectral representation that any Hermitian operator corresponds to a diagonal matrix $\Omega_{mn} = \omega_m\delta_{m,n}$ when represented in the basis of its own eigenstates. For this reason, the process of finding the eigenvalues and eigenfunctions of an operator (or matrix) is often referred to as *diagonalizing* the operator (or matrix).