PHY 4604 Spring 2013 – Homework 6

Due at 4:00 p.m. on Monday, April 8. Please turn in your homework in class before the deadline, or else bring it to NPB 2162. (You may push it under the door if NPB 2162 is unoccupied.) No credit will be available for homework submitted after the start of class on Friday, April 12.

Answer all four questions. Please write neatly and include your name on the front page of your answers. You must also clearly identify all your collaborators on this assignment. To gain maximum credit you should explain your reasoning and show all working.

1. *Compatible observables.* Two observables Ω and Λ are said to be "compatible" if the corresponding operators commute, i.e., $[\hat{\Omega}, \hat{\Lambda}] = 0$. Compatible observables can be chosen to have a complete set of simultaneous eigenstates $\{|\omega, \lambda\rangle\}$ such that

$$\hat{\Omega}|\omega, \lambda\rangle = \omega|\omega, \lambda\rangle \quad \text{and} \quad \hat{\Lambda}|\omega, \lambda\rangle = \lambda|\omega, \lambda\rangle.$$

Example (based on Problem 1-23 in *Modern Quantum Mechanics* by J. J. Sakurai): Consider a three-dimensional Hilbert space. In a certain basis, operators $\hat{\Omega}$ and $\hat{\Lambda}$ are represented by

$$\hat{\Omega} = \begin{pmatrix} a & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & -a \end{pmatrix}, \qquad \hat{\Lambda} = \begin{pmatrix} b & 0 & 0 \\ 0 & 0 & -ib \\ 0 & ib & 0 \end{pmatrix},$$

where a and b are both real.

   (a) Show that $\hat{\Omega}$ and $\hat{\Lambda}$ commute.

   (b) Find the spectrum of $\hat{\Omega}$ and the spectrum of $\hat{\Lambda}$.

   (c) Find a complete orthonormal set of simultaneous eigenstates of $\hat{\Omega}$ and $\hat{\Lambda}$. Label each eigenstate according to its eigenvalues of $\hat{\Omega}$ and $\hat{\Lambda}$.

2. *Sequential measurements.* Solve Problem 3.27 of Griffiths (2nd ed.).

Supplemental information (not needed to answer the question): The problem implicitly assumes that the measurements of A and B are ideal "measurements of the first kind" that leave the system after the measurement in one of the eigenstates of the measuring operator. The next paragraphs explain how the problem may be understood within the two different interpretations of quantum mechanics discussed in class.

In the *orthodox interpretation* of quantum mechanics, the state vector $|\Psi\rangle$ describes a single system (the "object"), and all measurements are postulated to be of the first kind. The act of measuring A cause a discontinuous reduction of the state vector from $|\Psi\rangle$ before the measurement to the eigenvector $|a\rangle$ corresponding to the observed eigenvalue a. This is presumably what the author of the problem had in mind.

One can also obtain the same answers to the three parts of this problem using an *empiricist ensemble* interpretation of quantum mechanics. Here, $|\Psi\rangle$ describes a virtual (i.e., imaginary) ensemble of objects subjected to the same preparation procedure. During a measurement, a combined state vector describing both the object and the measuring apparatus evolves continuously according to the Schrödinger equation, leaving the object in a final state that is generally not an eigenvector of the measuring operator.

In the empiricist ensemble interpretation, a measurement of the first kind can be regarded as an additional step in the preparation procedure of the ensemble describing the system after the measurement. An ensemble of objects prepared in state $|\Psi\rangle$ and then subjected to a measurement of A can be divided after the measurement into two sub-ensembles, one consisting of objects found to have the value $A = a_1$, the other consisting of objects found to have the value $A = a_2$. If we wish to make predictions concerning the subsequent behavior of all members of the original ensemble, then we should use the time-evolved version of the state vector $|\Psi\rangle$ that described the ensemble immediately before the measurement. However, if we wish to make predictions for the subsequent behavior of only those objects in the sub-ensemble that were found to have $A = a_1$, then we should use the time-evolved version of the state vector $|a_1\rangle$, since $|a_1\rangle$ describes the ensemble of objects subjected to the preparation procedure for $|\Psi\rangle$ followed by the additional preparation step of being selected for having the value $A = a_1$. Within this interpretation, the state vector is effectively reduced by taking into account the additional information obtained during the measurement, but we don't regard this reduction as a real physical process.

3. *Position and momentum wave functions.* A particle of mass m moves in the potential $V(x) = -Fx$, where F is a real constant with the dimensions of a force. Solution of the time-independent Schrödinger equation $\hat{H}|\psi_E\rangle = E|\psi_E\rangle$ in the coordinate representation leads to stationary-state wave functions $\psi_E(x) = \langle x|\psi_E\rangle$ that must be expressed in terms of Airy functions. This question asks you instead to perform the simpler task of finding the momentum-representation stationary states $\phi_E(p) = \langle p|\psi_E\rangle$.

   (a) Re-express $\hat{H}|\psi_E\rangle = E|\psi_E\rangle$ as a differential equation for $\phi_E(p)$.

   (b) Divide both sides of the equation you obtained in (a) by $\phi_E(p)$, and hence obtain a separable differential equation for $\ln \phi_E$ as function of p. Note: $d(\ln f) = (df)/f$.

   (c) Solve the differential equation you derived in (b) to express ϕ_E in terms of p, constants, and one unknown amplitude: $\phi_E(0)$.

   (d) Express the stationary-state wave function $\psi_E(x)$ as a momentum integral involving variables appearing previously in the question. Do not evaluate the integral.

4. *The infinite square well in three dimensions.* Consider a particle of mass m confined by hard walls to the region of space $0 < x < a_x$, $0 < y < a_y$, $0 < z < a_z$, inside which the potential is zero.

   (a) Write down the form of the stationary states and their energies.

   (b) Consider the particular case $a_x = a$, $a_y = a_z = a/\sqrt{2}$. List each energy eigenvalue E that satisfies $E < 10\pi^2\hbar^2/(2ma^2)$, along with that eigenvalue's degeneracy.