PHY 6645 Fall 2001 – Mid-Term Exam 1

Instructions: Attempt both Question 1 (worth 40 points) and Question 2 (worth 60 points). (Note that Question 2 continues on the back of this page.) The maximum score for each part of each question is shown in square brackets. To gain full credit you should explain your reasoning and show all working. Please write neatly and remember to include your name on the front page of your answers.

This is a closed-book exam. You are not allowed to consult any books, notes, or other papers, or to communicate with anyone other than the proctor. In accordance with the UF Honor Code, by turning in this exam to be graded, you affirm the following pledge: *On my honor, I have neither given nor received unauthorized aid in doing this assignment.*

1. A quantum mechanical system is described by a two-dimensional vector space spanned by orthonormal basis vectors $|1\rangle$ and $|2\rangle$. The Hamiltonian for this system is

$$H = \Delta \left(|1\rangle\langle 2| + |2\rangle\langle 1| \right).$$

We will also consider the operator

$$\Omega = \omega_1 |1\rangle\langle 1| + \omega_2 |2\rangle\langle 2|.$$

(a) What condition must the scalar Δ satisfy? [2]

(b) Provide the matrix representations of H and Ω in the basis $(|1\rangle, |2\rangle)$. [4]

(c) Find the eigenvalues and normalized eigenkets of H. [10]

(d) Provide the matrix representations of H and Ω in the eigenbasis of H. [8]

(e) Suppose that the state vector at time 0 is $|\psi(0)\rangle = |1\rangle$. Write down the state vector at an arbitrary time t, $|\psi(t)\rangle$. Give your answer in the basis $(|1\rangle, |2\rangle)$. [8]

(f) Given that the system is in state $|1\rangle$ at time 0, find the expectation value $\langle \Omega \rangle$ at time t. [8]

2. Consider a particle of mass m moving in one dimension according to the quantum mechanical Hamiltonian $H = P^2/2m + V(X)$, where

$$V(x) = \begin{cases} V_0 & x \le -a, \\ 0 & -a < x < a, \\ 2V_0 & x \ge a, \end{cases}$$

where $V_0 > 0$.

(a) List the range(s) of energies E within which you expect the energy eigenstates to be (i) forbidden; (ii) allowed at discrete energies only; (iii) allowed at all energies within the range. Within which range(s), if any, will the energy eigenstates be doubly degenerate? [8]

(b) The ground state (i.e., the lowest-energy eigenstate) of this system has energy E_0 and a wave function that may be written (up to an overall multiplicative constant) as

$$\psi_0(x) = \begin{cases} Be^{\kappa x} & x < -a, \\ \cos(kx + \phi) & -a < x < a, \\ Ae^{-\kappa' x} & x > a. \end{cases}$$

Express k, κ, and κ' as functions of E_0 and of parameters entering the Hamiltonian. [6]

(c) Apply the boundary conditions on the wave function $\psi_0(x)$ at $x = \pm a$ to obtain four equations connecting A, B, and ϕ. [12]

(d) Calculate $\psi_0(a)/\psi_0(-a)$ and $\tilde{\psi}_0(a)/\tilde{\psi}_0(-a)$, where $\tilde{\psi}_0(x) = d\ln\psi_0(x')/dx'|_{x'=x}$. Your answers should not contain A, B, or ϕ. [12]

(e) At what x value(s) will the point(s) of inflection of $\psi_0(x)$ be located? (At a point of inflection, $d^2\psi_0/dx^2 = 0$.) [4]

(f) Sketch the behavior of $|\psi_0(x)|$ as a function of x. The sketch should reflect, at least qualitatively, your answers to parts (d) and (e). Also, make sure that you show whether the maximum in $|\psi_0(x)|$ is located at $x > 0$, $x < 0$, or $x = 0$. [10]

(g) Eliminate A, B, and ϕ from the equations obtained in part (c) to obtain an equation connecting k, κ, and κ'. (Do not attempt to solve this equation.) [8]