PHY 6645 Fall 2001 – Homework 6

Due at the start of class on Thursday, October 25. No credit will be available for solutions submitted after 4 p.m. on Friday, October 26.

Answer all questions. To gain full credit you should explain your reasoning and show all working. Please write neatly and include your name on the front page of your answers.

1. Starting from

$$\frac{d}{dt}\langle\Omega\rangle = \frac{i}{\hbar}\langle[H,\Omega]\rangle + \left\langle\frac{\partial\Omega}{\partial t}\right\rangle,$$

and defining the operator

$$J = \{X, P\} - 2\langle X\rangle\langle P\rangle,$$

where $\{X, P\}$ is the anticommutator of X and P, show that the following equations are valid for the time evolution of any state of a quantum harmonic oscillator:

$$\begin{aligned}
\frac{d}{dt}\langle X\rangle &= \frac{1}{m}\langle P\rangle, \\
\frac{d}{dt}\langle P\rangle &= -m\omega^2\langle X\rangle, \\
\frac{d}{dt}\langle X^2\rangle &= \frac{1}{m}\langle\{X, P\}\rangle, \\
\frac{d}{dt}\langle P^2\rangle &= -m\omega^2\langle\{X, P\}\rangle, \\
\frac{d}{dt}(\Delta X)^2 &= \frac{1}{m}\langle J\rangle, \\
\frac{d}{dt}(\Delta P)^2 &= -m\omega^2\langle J\rangle, \\
\frac{d}{dt}\langle J\rangle &= \frac{2}{m}(\Delta P)^2 - 2m\omega^2(\Delta X)^2.
\end{aligned}$$

Note that the first two properties correspond to the classical equations of motion for the dynamical variables x and p.

2. Consider a harmonic oscillator wave function that at time $t = 0$ has the form of a Gaussian wave packet,

$$\psi(x, 0) = \frac{e^{i\theta}}{(2\pi w^2)^{1/4}} \exp\left[\frac{ip_0(x - x_0)}{\hbar} - \left(\frac{x - x_0}{2w}\right)^2\right],$$

where w, θ, x_0, and p_0 are all real parameters.

   (a) Evaluate $\langle X\rangle$, $\langle P\rangle$, ΔX, and ΔP in the state $\psi(x, 0)$, and confirm that it is a minimum-uncertainty state.

   (b) Evaluate $d^n(\Delta X)^2/dt^n$ and $d^n(\Delta P)^2/dt^n$ in the state $\psi(x, 0)$, where n is a positive integer. Hint: Apply some of the equations that you proved in the previous problem.

(c) Use your results from part (b) to show that the state $\psi(x,t)$ that evolves from $\psi(x,0)$ is also a minimum uncertainty state *provided that* $w = \sqrt{\hbar/2m\omega}$.

3. Consider a *coherent state* of the harmonic oscillator,

$$|z\rangle = \exp(za^\dagger - \tfrac{1}{2}|z|^2)|0\rangle,$$

where $|0\rangle$ is the ground state, $a^\dagger$ is the creation operator, and z is any complex number. Coherent states have many remarkable properties, but this question focuses on their position and momentum uncertainties.

(a) Express $|z\rangle$ as a sum over all eigenstates $|n\rangle$ of the harmonic oscillator, and confirm that $|z\rangle$ is normalized to unity.

(b) Suppose that a harmonic oscillator is initially in state $|\psi(0)\rangle = |z\rangle$. Using the representation of $|z\rangle$ derived in part (a), find $|\psi(t)\rangle$. Simplify your answer as much as possible.

(c) Evaluate $\langle X\rangle$, $\langle P\rangle$, ΔX, and ΔP in the state $|z\rangle$. Use the number basis (rather than the position basis) to obtain your answers.

(d) Demonstrate that $|z\rangle$ is an eigenket of the annihilation operator a. Use this fact to derive the wave function $\psi_z(x) = \langle x|z\rangle$. Show that your result is a special case of the initial wave function of the previous problem.

4. Shankar Exercise 9.4.4.

5. Consider the operators L_x, L_y, and L_z defined on Shankar Exercise 4.2.1.

(a) Show that these operators satisfy the commutation relations

$$[L_i, L_j] = i\varepsilon_{ijk}L_k,$$

where ε_{ijk} is the Levi-Civita symbol (defined, e.g., on page 319 of Shankar).

(b) Calculate ΔL_x, ΔL_y, and ΔL_z in the states

$$|\psi_1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad |\psi_2\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

(c) Calculate $\Delta L_x \Delta L_y$ in the states $|\psi_1\rangle$ and $|\psi_2\rangle$ defined above, and check that the appropriate uncertainty relation $\Delta\Omega\,\Delta\Lambda \geq \frac{1}{2}|\langle[\Omega,\Lambda]\rangle|$ is obeyed in each case.