

PHY 6645 Fall 2001 – Homework 9

Due at the start of class on Thursday, November 29. No credit will be available for solutions submitted after 4 p.m. on Friday, November 30.

Answer all questions. To gain full credit you should explain your reasoning and show all working. Please write neatly and include your name on the front page of your answers.

1. Shankar Exercise 12.5.3.
2. Shankar Exercise 12.5.7. (You may want to read through Ex. 12.5.6 to understand the motivation for Ex. 12.5.7.)
3. Shankar Exercise 12.5.12.
4. Consider a particle of mass  $m$  in a spherical box of radius  $a$ , i.e., a particle in a potential

$$V(\mathbf{r}) = \begin{cases} 0 & |\mathbf{r}| < a, \\ \infty & |\mathbf{r}| \geq a. \end{cases}$$

- (a) Find the ten lowest-energy eigenstates of this problem. For each eigenstate, list the quantum numbers, the energy in units of  $\hbar^2/ma^2$ , and the wave function in terms of spherical Bessel functions and spherical harmonics. (Do not bother to normalize the wave functions.)
- (b) Calculate the ten lowest-energy states for a particle in a cubic box that has the same volume as the spherical box. Compare the energies and degeneracies of the spherical and cubic spectra side-by-side.

Hint: You will need to look up the positions  $x$  of the first few zeros of  $j_l(x)$ . These can be found in standard references, e.g., M. Abramowitz and I. E. Stegun, *Handbook of Mathematical Functions* (Dover Publications, New York, 1974).

5. Modified from Ballentine Problem 10.7: Consider the following argument: For any Hamiltonian  $H = \mathbf{P} \cdot \mathbf{P}/2m + V(\mathbf{r})$ , it is clear that  $[H, \mathbf{R}] = -(i\hbar/m)\mathbf{P}$ . Therefore, in any eigenstate  $|\psi\rangle$  such that  $H|\psi\rangle = E|\psi\rangle$ ,

$$-\frac{i\hbar}{m}\langle \mathbf{P} \rangle = \langle \psi | [H, \mathbf{R}] | \psi \rangle = (\langle \psi | H) \mathbf{R} | \psi \rangle - \langle \psi | \mathbf{R} (H | \psi \rangle) = \langle \psi | E \mathbf{R} | \psi \rangle - \langle \psi | \mathbf{R} E | \psi \rangle = 0.$$

This argument, if correct, would establish that  $\langle \mathbf{P} \rangle = 0$  in any stationary state  $|\psi\rangle$ . However,  $V(\mathbf{r}) = 0$ ,  $\psi(\mathbf{r}) = \exp(i\mathbf{k} \cdot \mathbf{r})$  provides a counterexample.

Show that the argument presented above is valid only for bound states.

Hint: In lectures, the quantization of the eigenvalues of  $L_z$  ( $l_z = m\hbar$ ,  $m$  an integer) was argued based on single-valuedness of the wave function at the origin. Shankar (pp. 313 and 314) gives an alternative argument based on the requirement that  $L_z$  be Hermitian.

6. Shankar Exercises 13.1.1 and 13.3.3.
7. Shankar Exercise 13.4.1.

8. Merzbacher Problem 12.7: Compute the probability that the electron in a hydrogen atom will be found at a distance from the nucleus greater than its energy (the quantum mechanical value) would permit in classical theory. Perform this calculation for the  $n = 1$  and  $n = 2$  levels.
9. Modified from Merzbacher Problem 12.11:
- (a) Starting with the radial Schrödinger equation for the hydrogen atom [Shankar (13.1.2)], show that the transformation  $r = \alpha \bar{r}^2$ ,  $U(r) = \sqrt{\bar{r}} \bar{U}(\bar{r})$  produces an equation for  $\bar{U}(\bar{r})$  that, with appropriate choices of constants, is equivalent to the radial equation for an isotropic harmonic oscillator [Shankar (12.6.44)].
  - (b) Show the relation between the energy eigenvalues of the two problems.
  - (c) Show the relation between the quantum numbers  $n$  and  $l$  of the two problems.
  - (d) Express the radial wave function of the isotropic harmonic oscillator in the state having quantum numbers  $n$  and  $l$  as an associated Laguerre polynomial with half-integer indices.