

## One-Dimensional Wave Mechanics

General arguments and qualitative results will be presented in class concerning solutions to Schrödinger's wave equation for one-dimensional systems. In order to understand and apply this discussion, you need to be able to carry out detailed calculations for a number of standard examples, each described by a simple, piecewise-constant potential. You should make sure that you can do all the exercises below. If you get stuck, feel free to consult standard texts, other students, and/or me.

The potentials below are listed in order of increasing complexity. For each one, you should be able to

- Identify the range of energies  $E$  over which eigenstates of the time-independent Hamiltonian will be (i) continuously distributed in energy; (ii) forbidden on general principle; and (iii) not ruled out, but if present distributed with discrete eigenvalues.
- Determine the degeneracy with which any energy  $E$  would appear, should it turn out to be an allowed eigenvalue.
- Set up the form of a general solution  $\psi_E(x)$  of the Schrödinger equation within each region of constant potential.
- Apply the appropriate boundary conditions at each point where the potential changes.
- Where relevant, formulate the equation satisfied by bound-state solutions. (It is not always straightforward to solve this equation.)
- Where relevant, construct (i) the wavefunction  $\psi_{E,R}(x)$  describing a right-moving wave coming from  $x = -\infty$ , plus all scattered waves; and/or (ii) the wavefunction  $\psi_{E,L}(x)$  corresponding to a left-moving wave coming from  $x = +\infty$ , plus all scattered waves. Be able to figure out (qualitatively at least) the behavior when a wave packet approaches from either  $x = -\infty$  or  $x = +\infty$ .
- Calculate the probability current density  $j$  at any point  $x$ , and where relevant, calculate reflection and transmission coefficients ( $R$  and  $T$ ) for waves incoming from  $x = -\infty$  or  $x = +\infty$ .

1. **Step Potential:**  $V(x) = V_0\theta(x)$  with  $V_0 > 0$ .

- (a) Construct explicitly the states  $\psi_{E,R}(x)$  and  $\psi_{E,L}(x)$  (where they exist) and calculate  $j$ ,  $R$ , and  $T$  for each state.
- (b) Verify that  $T(E)$  is independent of the direction of incidence. Can you prove that this is true for any one-dimensional system described by a real Hamiltonian?
- (c) In situations where both  $\psi_{E,L}(x)$  and  $\psi_{E,R}(x)$  exist, are these two states orthogonal? If not, construct an orthogonalized pair of states to replace them.

- (d) Check that you understand what happens when a Gaussian wave packet scatters from the step. Shankar has a good discussion of this.
- (e) Do Shankar Ex. 5.3.4, showing that  $j(x) = j_A(x) + j_B(x)$  in any region of constant potential  $V < E$ .

**2. Rectangular Potential Barrier:**  $V(x) = V_0\theta(a - |x|)$  with  $V_0, a > 0$ .

- (a) Construct the transfer matrix for this potential by combining the transfer matrices for two step potentials.
- (b) Construct explicitly the states  $\psi_{E,R}(x)$  and  $\psi_{E,L}(x)$  (where they exist) and calculate  $j$ ,  $R$ , and  $T$  for each state. It is of interest to look for resonances, i.e., peaks in the transmission as a function of  $E$ .
- (c) Analyze the behavior of the eigensolutions in two special cases: (i) the infinite potential barrier,  $V_0 = \infty$ ; (ii) the delta-function barrier,  $V(x) = 2V_0a\delta(x)$ .

**3. Rectangular Potential Well:**  $V(x) = -V_0\theta(a - |x|)$  with  $V_0, a > 0$ .

- (a) Construct the states  $\psi_{E,R}(x)$  and  $\psi_{E,L}(x)$  for arbitrary  $E > 0$ , and calculate  $j$ ,  $R$ , and  $T$  for each state. Again, it is interesting to look for resonances in  $T(E)$ . Merzbacher describes this in some detail.
- (b) Formulate the constraint on the allowed eigensolutions for  $E < 0$ .
- (c) Check that you can find the  $E < 0$  solutions explicitly in two special cases: (i) the particle in a box problem,  $V(x) = 0$  for  $|x| < a$ ,  $V(x) = +\infty$  for  $|x| \geq a$ ; (ii) the delta-function well,  $V(x) = -2V_0a\delta(x)$ .
- (d) Work through the graphical procedure for finding allowed eigensolutions for the general rectangular well. (This is problem 3 of Homework 4.)
- (e) Do Shankar Ex. 5.2.2, proving that every attractive potential in one dimension has at least one bound state. (We will see later that this need not be the case in higher dimensions.)

**4. Double Barriers:**  $V(x) = V_0\theta(|x| - a)\theta(b - |x|)$  with  $V_0 > 0$  and  $b > a > 0$ .

- (a) Construct the transfer matrix element for this potential by combining the transfer matrices for two rectangular barriers.
- (b) Plot the transmission coefficient  $T$  vs  $E/V_0$  for  $a = \sqrt{18\hbar^2/mV_0}$  and (i)  $b = 1.2a$ ; (ii)  $b = 3a$ ; (iii)  $b = 10a$ . It will suffice to calculate  $T(E)$  numerically.