PHY 6645 Fall 2002 – Mid-Term Exam 1

Instructions: Attempt both questions, each of which is worth 50 points. The maximum score for each part of each question is shown in square brackets. To gain full credit you should explain your reasoning and show all working. Please write neatly and remember to include your name on the front page of your answers.

Please read carefully: During this exam, you may use Shankar's *Principles of Quantum Mechanics* and lecture notes from this course. You are not permitted to consult any other books, notes, or papers, or to communicate with anyone other than the proctor. In accordance with the UF Honor Code, by turning in this exam to be graded, you affirm the following pledge: *On my honor, I have neither given nor received unauthorized aid in doing this assignment.*

1. A quantum mechanical system is described by a two-dimensional vector space spanned by orthonormal basis vectors $|1\rangle$ and $|2\rangle$. These basis vectors are the eigenvectors of an operator Λ such that
$$\Lambda|j\rangle = \lambda_j|j\rangle \quad \text{for } j = 1, 2.$$
The Hamiltonian for this system can be written
$$H = \alpha\left(|1\rangle\langle 1| + |2\rangle\langle 2|\right) + 2\alpha\left(|1\rangle\langle 2| + |2\rangle\langle 1|\right).$$
At time $t = 0$, the system is described by the state operator
$$\rho(0) = a|1\rangle\langle 1| + b|1\rangle\langle 2| + c|2\rangle\langle 1|.$$
   - (a) Find the matrix representation of H in the basis $(|1\rangle, |2\rangle)$. [3]
   - (b) Find the eigenvalues of H, and the normalized eigenkets in the basis $(|1\rangle, |2\rangle)$. [8]
   - (c) Find the matrix representation of the propagator $U(t)$ in the basis $(|1\rangle, |2\rangle)$. [12]
   - (d) What constraints must be satisfied by the coefficients a, b, and c entering $\rho(0)$? Simplify $\rho(0)$ by eliminating as many as possible of these coefficients, and use this simplified $\rho(0)$ throughout the rest of the question. [6]
   - (e) Find the matrix representation of $\rho(0)$ in the basis $(|1\rangle, |2\rangle)$. [3]
   - (f) Calculate matrix representation of $\rho(t)$ in the basis $(|1\rangle, |2\rangle)$. [12]
   - (g) What is the probability that a measurement of Λ at time t will yield the result λ_1? [6]

2. Consider a particle of mass m moving in one dimension under the influence of a double-step potential. The quantum mechanical Hamiltonian for this system is $H = P^2/2m + V(X)$, where

$$V(x) = \begin{cases} -V_0 & x \leq -a, \\ 0 & -a < x < a, \\ V_0 & x \geq a, \end{cases}$$

with $V_0 = \pi^2\hbar^2/ma^2$.

(a) List the range(s) of energies E (if any) within which you expect the energy eigenstates to be (i) forbidden; (ii) allowed at discrete energies only; (iii) allowed at all energies within the range; (iv) doubly degenerate. [8]

The remainder of this problem concerns a particular stationary state of this problem: one of energy $E = V_0/2$, having a wave function $\psi(x)$.

(b) Write down the form of $\psi(x)$ in the region $x \leq -a$. Express the wavelength or exponential decay length of $\psi(x)$ (whichever is appropriate) as a multiple of a. (The exponential decay length is l in $e^{-x/l}$.) [6]

(c) Write down the form of $\psi(x)$ in the region $-a \leq x \leq a$. Express the wavelength or exponential decay length of $\psi(x)$ in this region as a multiple of a. [6]

(d) Write down the form of $\psi(x)$ in the region $x \geq a$. Express the wavelength or exponential decay length of $\psi(x)$ in this region as a multiple of a. [6]

(e) By examining the boundary conditions at $x = -a$, determine whether the maximum amplitude of $\psi(x)$ in the region $x \leq -a$ is greater than, less than, or equal to the maximum amplitude of $\psi(x)$ in the region $-a \leq x \leq a$. [12]

(f) Sketch a graph of the probability density $|\psi(x)|^2$ as a function of x. The horizontal axis should range from $x = -3a$ to $x = 3a$, and should have the points $x = -a$ and $x = a$ explicitly labeled.

You should be careful to represent correctly as many features of the graph as possible (based on the information derived in the previous parts of the problem). [12]