Addition of Two Arbitrary Angular Momenta

- We are going to consider the general problem of adding two independent angular momenta $\mathbf{J}_1$ and $\mathbf{J}_2$, e.g., the orbital and spin angular momenta of a single particle, or the spin of two electrons. In many cases, the magnitude of each of the angular momenta (J_1^2 and J_2^2) can be considered to be fixed, but their z components ($J_{1,z}$ and $J_{2,z}$) can vary. The goal is to construct composite states that are simultaneous eigenstates of
$$J^2 = |\mathbf{J}_1 + \mathbf{J}_2|^2$$
and
$$J_z = J_{1,z} + J_{2,z}. \tag{1}$$

- Assume that we start with two sets of angular momentum eigenkets, $\{|j_k, m_k\rangle, m_k = -j_k, -j_k+1, \ldots, j_k-1, j_k\}$ $(k = 1, 2)$, each set exhibiting the standard properties
$$\begin{aligned} J_k^2 \, |j_k, m_k\rangle &= j_k(j_k+1)\,\hbar^2\,|j_k, m_k\rangle, \\ J_{k,z}\,|j_k, m_k\rangle &= m_k\,\hbar\,|j_k, m_k\rangle, \\ J_{k,\pm}\,|j_k, m_k\rangle &= c_\pm(j_k, m_k)\,\hbar\,|j_k, m_k \pm 1\rangle. \end{aligned}$$
where
$$c_\pm(j, m) = {}^{+}\!\sqrt{(j \mp m)(j \pm m+1)}.$$
Since the operators $\mathbf{J}_k$ obey the commutation relations $[J_j^2, J_{k,l}] = 0$, $[J_{1,l}\,, J_{2,m}] = 0$, and $[J_{k,l}\,, J_{k,m}] = i\hbar\epsilon_{lmn}J_{k,n}$ (for $l, m, n \in \{x, y, z\}$ and $j, k \in \{1, 2\}$), the direct product basis kets
$$|j_1, m_1; j_2, m_2\rangle = |j_1, m_1\rangle \otimes |j_2, m_2\rangle$$
are simultaneous eigenkets of J_1^2, $J_{1,z}$, J_2^2, $J_{2,z}$, and J_z, but *not* (in general) of
$$J^2 \equiv J_1^2 + J_2^2 + J_{1,+}J_{2,-} + J_{1,-}J_{2,+} + 2J_{1,z}J_{2,z}. \tag{2}$$

- The commutation relations above imply that the eigenstates of J^2 and J_z are also eigenstates of J_1^2 and J_2^2, but not necessarily eigenstates of $J_{1,z}$ or $J_{2,z}$. Accordingly, we denote these states by $|j, m, j_1, j_2\rangle$, or often just $|j, m\rangle$.

- In principle, one can find the states $|j, m\rangle$ by diagonalizing the matrix representation of J^2 in the direct product basis $|j_1, m_1; j_2, m_2\rangle$. However, the dimension of that basis is $(2j_1 + 1)(2j_2 + 1)$, so this brute-force approach is cumbersome for large j_1 and/or j_2.

- There is a more efficient procedure for systematically generating the states $|j, m\rangle$ by making use of the operators $J_\pm = J_x \pm iJ_y$:

  1. We start with the state
  $$|j_1{+}j_2, j_1{+}j_2\rangle = |j_1, m_1; j_2, m_2\rangle \equiv |j_1, m_1\rangle \otimes |j_2, m_2\rangle, \tag{3}$$
  which has the largest possible value of m that can be constructed from the two sets of angular momentum states. It is simple to verify using Eqs. (1) and (2) that this is an eigenstate of both J^2 and J_z.

2. Since Eq. (3) describes an angular momentum eigenstate, we can generate another such eigenstate by acting with J_- on both sides of the equation:

$$\begin{aligned} &c_-(j_1+j_2, j_1+j_2)|j_1+j_2, j_1+j_2-1\rangle = \\ &\qquad c_-(j_1, j_1)|j_1, j_1-1; j_2, j_2\rangle + c_-(j_2, j_2)|j_1, j_1; j_2, j_2-1\rangle. \end{aligned} \tag{4}$$

   Dividing through by $c_-(j_1+j_2, j_1+j_2)$, we can extract $|j_1+j_2, j_1+j_2-1\rangle$.

3. By applying J_- a total of $2(j_1+j_2)$ times, we can extract a complete multiplet of states of total angular momentum $j = j_1+j_2$.

4. Now we consider the subspace of direct product states having $J_z = (j_1+j_2-1)\hbar$. There are exactly two such states: $|j_1, j_1-1; j_2, j_2\rangle$ and $|j_1, j_1; j_2, j_2-1\rangle$. Since we singled out one linear combination of these states in Eq. (4), we can construct a state

$$|j_1+j_2-1, j_1+j_2-1\rangle = A\,[c_-(j_2, j_2))|j_1, j_1-1; j_2, j_2\rangle - c_-(j_1, j_1)|j_1, j_1; j_2, j_2-1\rangle] \tag{5}$$

   that (i) is orthogonal to $|j_1+j_2, j_1+j_2-1\rangle$ defined in Eq. (4), (ii) satisfies

$$J_z|j_1+j_2-1, j_1+j_2-1\rangle = (j_1+j_2-1)\hbar|j_1+j_2-1, j_1+j_2-1\rangle$$

   and

$$J_+|j_1+j_2-1, j_1+j_2-1\rangle = 0,$$

   and (iii) is therefore an eigenstate of J^2 and J_z with total angular momentum $j = j_1 + j_2 - 1$. The prefactor A entering $|j_1+j_2-1, j_1+j_2-1\rangle$ is determined by the requirement of normalization and the sign convention that the coefficient of $|j_1, j_1; j_2, j_2-1\rangle$ is real and positive.

5. We now apply J_- repeatedly to Eq. (5) to generate a complete multiplet of angular momentum $j = j_1+j_2-1$.

6. In general, there will be three direct product states having $J_z = (j_1+j_2-2)\hbar$, from which two linear combinations have already been formed: $|j_1+j_2, j_1+j_2-2\rangle$ and $|j_1+j_2-1, j_1+j_2-2\rangle$. This leaves a third linear combination, which must be the top state of yet another multiplet, this one having total angular momentum $j = j_1+j_2-2$.

7. Each time we reduce m by one, we increase by one the number of direct product states having $J_z = m\hbar$. By forming a linear combination of these states that is orthogonal to all those of total angular momentum $j > m$ already constructed, we identify the top state of a multiplet of total angular momentum $j = m$. This state is uniquely determined by the requirements of normalization, and the convention that the coefficient of $|j_1, j_1; j_2, j-j_1\rangle$ is real and positive.

8. This pattern continues until we have formed a multiplet of $j = |j_1 - j_2|$.
   The subspace $J_z = |j_1 - j_2|\hbar$ contains the product state $|j_1, j_1\rangle \otimes |j_2, -j_2\rangle$ and/or $|j_1, -j_1\rangle \otimes |j_2, j_2\rangle$ (depending on whether j_1 is greater than, smaller than, or equal to j_2). Since $J_{k,-}\,|j_k, -j_k\rangle = 0$, we do not generate an additional direct product state having $J_z = m\hbar$ for any $m < |j_1 - j_2|$, and so we do not generate any multiplet of total angular momentum $j < |j_1-j_2|$.

9. The conclusion from this procedure is that states of angular momentum j_1 and j_2 can be combined to form eigenstates of total angular momentum j satisfying $|j_1-j_2| \le j \le j_1+j_2$, with exactly one multiplet possible for each possible j value, i.e.,
$$j_1 \otimes j_2 = j_1+j_2 \ \oplus \ j_1+j_2-1 \ \oplus \ \ldots \ \oplus \ |j_1-j_2|+1 \ \oplus \ |j_1-j_2|.$$

- The total angular momentum eigenkets can be conveniently written in the form

$$|j, m, j_1, j_2\rangle = \sum_{m_1=-j_1}^{+j_1} \sum_{m_2=-j_2}^{+j_2} |j_1, m_1; j_2, m_2\rangle\langle j_1, m_1; j_2, m_2|j, m, j_1, j_2\rangle, \tag{6}$$

where $\langle j_1, m_1; j_2, m_2|j, m, j_1, j_2\rangle \equiv \langle j_1, m_1; j_2, m_2|j, m\rangle$ is a **Clebsch-Gordan coefficient**.

- The Clebsch-Gordan coefficients have the following properties:

1. $\langle j_1, m_1; j_2, m_2|j, m\rangle \neq 0$ iff $j \in \{j_1+j_2,\ j_1+j_2-1, \ldots, |j_1-j_2|+1,\ |j_1-j_2|\}$ and $m = m_1 + m_2$.
2. Under the *Condon-Shortley* convention, all Clebsch-Gordan coefficients are real, and $\langle j_1, j_1; j_2, j-j_1|j, j\rangle$ is positive.
3. By applying $J_\pm$ to Eq. (6), one can obtain the recursion relations

$$\begin{aligned} c_\pm(j,m)\langle j_1, m_1; j_2, m_2|j, m \pm 1\rangle &= c_\mp(j_1, m_1)\langle j_1, m_1 \mp 1; j_2, m_2|j, m\rangle \\ &+ c_\mp(j_2, m_2)\langle j_1, m_1; j_2, m_2 \mp 1|j, m\rangle. \end{aligned}$$

4. Taking the norm of each side of Eq. (6) gives the normalization

$$\sum_{m_1,m_2} |\langle j_1, m_1; j_2, m_2|j, m\rangle|^2 = 1.$$

5. The Clebsch-Gordan coefficients are orthogonal, in the sense that

$$\sum_{m_1,m_2} \langle j_1, m_1; j_2, m_2|j, m\rangle\langle j_1, m_1; j_2, m_2|j', m'\rangle = \delta_{j,j'}\delta_{m,m'}$$

and

$$\sum_{j,m} \langle j_1, m_1; j_2, m_2|j, m\rangle\langle j_1, m_1'; j_2, m_2'|j, m\rangle = \delta_{m_1,m_1'}\delta_{m_2,m_2'}.$$

6. Finally, it can be shown that

$$\begin{aligned} \langle j_1, m_1; j_2, m_2|j, m\rangle &= (-1)^{j_1+j_2-j}\langle j_1, -m_1; j_2, -m_2|j, -m\rangle \\ &= \langle j_2, -m_2; j_1, -m_1|j, -m\rangle. \end{aligned}$$

Properties (1)–(4) are sufficient to fully determine the Clebsch-Gordan coefficients. However, in practice, one usually looks up the numerical values of these coefficients from standard tables or software packages.