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# The Hamiltonian Formulation of Classical Mechanics: Key Ideas

1. The state of a system containing $n$ degrees of freedom (e.g., $N$ unconstrained particles in $d$ spatial dimensions correspond to $n = Nd$) is described by $2n$ canonical variables:

   - $n$ generalized coordinates, $\mathbf{q} = (q_1, q_2, \ldots, q_n)$.
   - $n$ conjugate momenta, $\mathbf{p} = (p_1, p_2, \ldots, p_n)$.

   N.B. The momentum conjugate to $q_i$ is formally defined in the Lagrangian picture: $p_i = \partial L(\mathbf{q}, \dot{\mathbf{q}}, t)/\partial \dot{q}_i$.

2. The dynamics of the system are governed by $2n$ first-order partial differential equations, *Hamilton's equations*:
$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}, \qquad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}.$$
   These equations replace the $n$ second-order PDE's corresponding to Newton's second law, $\mathbf{F} = m\mathbf{a}$, applied to each particle.

3. In many (but not all) situations, the *Hamiltonian* $H(\mathbf{q}, \mathbf{p}, t)$ is the total energy of the system.

4. The equation of motion of any *dynamical variable* $\omega(\mathbf{q}, \mathbf{p}, t)$ is
$$\frac{d\omega}{dt} = \{\omega, H\} + \frac{\partial \omega}{\partial t}, \tag{1}$$
   where the *Poisson bracket* is defined to be
$$\{\omega, \lambda\} = \sum_i \left( \frac{\partial \omega}{\partial q_i}\frac{\partial \lambda}{\partial p_i} - \frac{\partial \omega}{\partial p_i}\frac{\partial \lambda}{\partial q_i} \right).$$
   Hamiltonian's equations are a special case of Eq. (1):
$$\frac{dq_i}{dt} = \{q_i, H\}, \qquad \frac{dp_i}{dt} = \{p_i, H\}.$$
   Note also that
$$\{q_i, q_j\} = 0, \qquad \{p_i, p_j\} = 0, \qquad \{q_i, p_j\} = \delta_{ij}.$$

5. A dynamical variable $\omega$ that has no explicit time dependence ($\partial\omega/\partial t = 0$) and has a vanishing Poisson bracket with the Hamiltonian is a *conserved quantity.* The existence of conserved dynamical variables is intimately tied to symmetries of the Hamiltonian. Specifically, $\omega$ is conserved if $H$ is invariant under the infinitesimal transformation
$$q_i \to q_i + \varepsilon \frac{\partial \omega}{\partial p_i} \equiv q_i + \varepsilon\{q_i, \omega\}, \qquad p_i \to p_i - \varepsilon \frac{\partial \omega}{\partial q_i} \equiv p_i + \varepsilon\{p_i, \omega\}.$$