PHY 6645 Fall 2003 – Mid-Term Exam 1

Instructions: Attempt both question 1 (worth 45 points) and question 2 (over the page, worth 55 points). The maximum score for each part of each question is shown in square brackets. To gain full credit you should explain your reasoning and show all working. Please write neatly and remember to include your name on the front page of your answers.

Please read carefully: During this exam, you may use Shankar's *Principles of Quantum Mechanics* and lecture notes from this course. You may also use standard mathematical tables. You may quote without proof any results given in these sources; however, you should cite the source for the result (e.g., "Shankar page 25"). You are not permitted to consult any other books, notes, or papers, or to communicate with anyone other than the proctor. In accordance with the UF Honor Code, by turning in this exam to be graded, you affirm the following pledge: *On my honor, I have neither given nor received unauthorized aid in doing this assignment.*

1. A quantum mechanical system is described by a two-dimensional vector space spanned by orthonormal basis vectors $|1\rangle$ and $|2\rangle$. The Hamiltonian for this system is

$$H = \epsilon\left(-4|1\rangle\langle 1| + 4|2\rangle\langle 2| + 3|1\rangle\langle 2| + 3|2\rangle\langle 1|\right),$$

where $\epsilon > 0$. We will also consider the operator

$$\Lambda = \lambda_0(|1\rangle\langle 2| + |2\rangle\langle 1|).$$

(a) [4 points] Provide the matrix representations of H and Λ in the basis $\{|1\rangle, |2\rangle\}$.

(b) [8 points] Find the eigenvalues (E_1 and E_2, with $E_1 < E_2$) and normalized eigenkets ($|E_1\rangle$ and $|E_2\rangle$) of H in the basis $\{|1\rangle, |2\rangle\}$.

(c) [13 points] Find the matrix representation of the propagator $U(t)$ in the basis $\{|1\rangle, |2\rangle\}$.

(d) [20 points] Suppose that the state vector at time 0 is $|\psi(0)\rangle = |2\rangle$. The value of the dynamical variable λ corresponding to the operator Λ is measured at time $t > 0$. What are the possible measured values of λ and their respective probabilities $P_\lambda(t)$?

2. Consider a particle of mass m moving in one dimension under the influence of the potential

$$V(x) = \frac{\hbar^2 Q}{2m}\left[\delta(x+a) - \delta(x-a)\right],$$

where $\delta(x)$ is the Dirac delta function, and Q and a are positive, real quantities. Let $\psi_E(x)$ be the wave function describing a stationary state of energy E.

Let us first consider unbound solutions ($E > 0$) and write

$$\psi_E(x) = A_j e^{ikx} + B_j e^{-ikx}, \quad k = |\sqrt{2mE}|/\hbar,$$

where $j = 1$ describes $x \leq -a$, $j = 2$ describes $|x| \leq a$, and $j = 3$ describes $x \geq a$. Define the 2×2 transfer matrix $P_E(n,m)$ by the relation

$$\begin{pmatrix} A_n \\ B_n \end{pmatrix} = P_E(n,m)\begin{pmatrix} A_m \\ B_m \end{pmatrix}.$$

(a) [13 points] State the boundary conditions on the wave function at $x = -a$. By applying these boundary conditions, find the transfer matrix $P(2,1)$ in terms of variables introduced above.

(b) [8 points] Find the transfer matrix $P(3,2)$ in terms of variables introduced above. Hint: You can do this by applying boundary conditions at $x = a$, but you can obtain $P(3,2)$ more efficiently by suitably modifying $P(2,1)$.

(c) [4 points] Show that the bottom-right element of the overall transfer matrix $P \equiv P(3,1)$ is

$$P_{22} = 1 + \frac{Q^2}{4k^2}\left(1 - e^{i4ka}\right).$$

(d) [8 points] Argue that the transmission coefficient for this potential is $T = |P_{22}|^{-2}$. (Hint: Start with a general expression for T, valid for any real, piecewise-constant potential, and specialize to the problem at hand.) Hence, find T as a function of k, Q, and a.

(e) [6 points] Find the value(s) of k at which T takes its maximum value.

Let us now seek bound-state solutions $\psi_E(x)$ having $E < 0$.

(f) [8 points] Recalling that bound states occur when $P_{22} = 0$, obtain a condition relating the bound-state value(s) of $\kappa = |\sqrt{-2mE}|/\hbar$ to Q and a. Hint: You should be able to obtain P_{22} for $E < 0$ by suitably modifying the result of part (c).

(g) [8 points] Show that bound state solutions can exist only for $0 < \kappa < Q/2$.