PHY 6645 Fall 2003 – Mid-Term Exam 2

Instructions: Attempt both question 1 (worth 50 points) and question 2 (worth 50 points). The maximum score for each part of each question is shown in square brackets. To gain full credit you should explain your reasoning and show all working. Please write neatly and remember to include your name on the front page of your answers.

Please read carefully: During this exam, you may use Shankar's *Principles of Quantum Mechanics* and lecture notes from this course. You may also use standard mathematical tables. You may quote without proof any results given in these sources; however, you should cite the source for the result (e.g., "Shankar page 25"). You are not permitted to consult any other books, notes, or papers, or to communicate with anyone other than the proctor. In accordance with the UF Honor Code, by turning in this exam to be graded, you affirm the following pledge: *On my honor, I have neither given nor received unauthorized aid in doing this assignment.*

Do not turn the page until instructed to do so

1. An ion can sit inside a solid at two sites of equal energy, L (left) and R (right), represented by orthonormal kets $|L\rangle$ and $|R\rangle$ in a two-dimensional vector space. The ion's environment permits it to tunnel between these two sites. As a result, the ion can be described by an effective Hamiltonian $H = \epsilon I + \eta\,\Pi$, where I is the identity operator, $\Pi = |L\rangle\langle R| + |R\rangle\langle L|$ is the spatial inversion (or parity) operator, and $\eta > 0$. Another operator of interest is the electric dipole moment $D = qa(|R\rangle\langle R| - |L\rangle\langle L|)$, where q is the charge of the ion and a is the distance between sites L and R.

   (a) [15 points] Find the commutators $[H, \Pi]$, $[\Pi, D]$, and $[H, D]$. What can you deduce from these commutators about the eigenbases of H, D, and Π?

   (b) [5 points] Use your answer to part (a) to obtain a non-trivial uncertainty relation involving $\Delta E \equiv \Delta H$ and either ΔD or $\Delta\Pi$. (A non-trivial uncertainty relation is an operator inequality of the form $\Delta\Omega\,\Delta\Lambda \geq \Gamma$, where Γ is not identically zero.)

   (c) [10 points] Use your answer to part (a) to obtain a rigorous energy-time uncertainty relation of the form $\Delta E\,\Delta t \geq \hbar/2$. Make sure that you give a precise definition of Δt.

   (d) [20 points] Verify by explicit calculation that the uncertainty relation you deduced in part (b) is obeyed in the state described by the state operator

$$\rho = \frac{1}{3}|L\rangle\langle L| + \frac{2}{3}|R\rangle\langle R| + \frac{i}{6}\left(|L\rangle\langle R| - |R\rangle\langle L|\right).$$

2. Consider a one-dimensional harmonic oscillator of mass m and frequency ω.

   (a) [10 points] Calculate the position uncertainty ΔX in the number eigenstate $|n\rangle$ ($n = 0, 1, 2, \ldots$).

   (b) [15 points] Consider the family of states of the form $|\theta,\phi\rangle = \cos\theta\,|0\rangle + e^{i\phi}\sin\theta\,|2\rangle$, where θ and ϕ are real. Show that the position uncertainty in the state $|\theta,\phi\rangle$ is

$$\Delta X(\theta,\phi) = \left[\frac{\hbar}{2m\omega}\left(3 - 2\cos 2\theta + \sqrt{2}\sin 2\theta\,\cos\phi\right)\right]^{1/2}.$$

   (c) [10 points] $\Delta X(\theta,\phi)$ is minimized for $\phi = \phi_0 = \pi$ and $\theta = \theta_0 = \frac{1}{2}\tan^{-1}(1/\sqrt{2})$, where the inverse tangent is taken such that $0 < \theta_0 < \pi/2$. (You do not need to prove this.) Show that ΔX in the state $|\theta_0,\phi_0\rangle$ is smaller than its value in any stationary state of the harmonic oscillator.

   (d) [15 points] Suppose that the harmonic oscillator is in state $|\theta_0,\phi_0\rangle$ at time $t = 0$. Find ΔX at an arbitrary time $t > 0$. Show that there are times when ΔX is greater than the position uncertainty in at least one stationary state $|n\rangle$.