PHY 6645 Fall 2003

Information Concerning Mid-Term Exam 2

- The second mid-term exam will be held from 6:00 to 8:00 p.m. on Friday, November 21 in NPB 1101.
- The exam will cover all material discussed in the course up to and including the end of the discussion of time reversal symmetry (completed in class on Wednesday, November 12). A list of topics, cross referenced to the course texts, appears at www.phys.ufl.edu/~kevin/teaching/6645/03fall/topics.html. The focus will be on topics introduced since the first mid-term.
- The exam will test your ability to apply the concepts that have been introduced in the course. Parts of the exam will be related to the homework that you have been assigned, but the exam questions will generally be algebraically simpler. Other parts may address unfamiliar applications of course material.
- You will be allowed to use Shankar and your lecture notes during the exam.
- You may bring a calculator and mathematical tables into the exam. However, it is unlikely that you will need these aids.
- **You must not use any other written/printed materials during the exam.** For instance, homework solutions or worked problems are not permitted.
- **You must not consult any person other than the proctor during the exam.**

The remainder of this handout contains homework and examination questions from past years. These questions are provided for practice purposes only. They will not be collected for grading. Solutions to these questions are being distributed in class.

1. **Fall 2002 Mid-Term Exam 1, Question 1 (worth 50% of the exam)**

   A quantum mechanical system is described by a two-dimensional vector space spanned by orthonormal basis vectors $|1\rangle$ and $|2\rangle$. These basis vectors are the eigenvectors of an operator Λ such that
   $$\Lambda|j\rangle = \lambda_j|j\rangle \quad \text{for } j = 1, 2.$$
   The Hamiltonian for this system can be written
   $$H = \alpha\left(|1\rangle\langle 1| + |2\rangle\langle 2|\right) + 2\alpha\left(|1\rangle\langle 2| + |2\rangle\langle 1|\right).$$
   At time $t = 0$, the system is described by the state operator
   $$\rho(0) = a|1\rangle\langle 1| + b|1\rangle\langle 2| + c|2\rangle\langle 1|.$$

   (a) Find the matrix representation of H in the basis $(|1\rangle, |2\rangle)$. [3]

   (b) Find the eigenvalues of H, and the normalized eigenkets in the basis $(|1\rangle, |2\rangle)$. [8]

(c) Find the matrix representation of the propagator $U(t)$ in the basis $(|1\rangle, |2\rangle)$. [12]

(d) What constraints must be satisfied by the coefficients a, b, and c entering $\rho(0)$? Simplify $\rho(0)$ by eliminating as many as possible of these coefficients, and use this simplified $\rho(0)$ throughout the rest of the question. [6]

(e) Find the matrix representation of $\rho(0)$ in the basis $(|1\rangle, |2\rangle)$. [3]

(f) Calculate matrix representation of $\rho(t)$ in the basis $(|1\rangle, |2\rangle)$. [12]

(g) What is the probability that a measurement of Λ at time t will yield the result λ_1? [6]

2. **Fall 2001 Mid-Term Exam 2, Question 1 (worth 65% of the exam)**

This question concerns a one-dimensional harmonic operator described by the Hamiltonian $H = P^2/2m + m\omega^2 X^2/2$, which has eigenkets $|n\rangle$ with energies $(n + \frac{1}{2})\hbar\omega$ ($n = 0$, 1, 2, ...). Use the **Heisenberg picture** in answering all parts of the question.

(a) Starting from the Heisenberg equation, $d\Omega/dt = (i/\hbar)[H, \Omega] + \partial\Omega/\partial t$, derive equations of motion for the position operator X and the momentum operator P. (These equations should contain no commutators.) [15]

(b) Show that the lowering (annihilation) operator a obeys an equation of motion of the form $da/dt \propto a$. Integrate this equation of motion to obtain $a(t)$. [10]

(c) Let $|\phi\rangle = c_0|0\rangle + c_1|1\rangle$ be the normalized linear combination of $|0\rangle$ and $|1\rangle$ that has the greatest value of $\langle P(t = 0)\rangle$. Find the coefficients c_0 and c_1. [20]

(d) Find $\langle P(t)\rangle$ for the state $|\phi\rangle$. Eliminate c_0 and c_1 from your final answer. [10]

(e) Find $\langle P(t)\rangle$ for the mixed state described by the state operator

$$\rho = |c_0|^2|0\rangle\langle 0| + |c_1|^2|1\rangle\langle 1|,$$

where c_0 and c_1 are the coefficients defined in part (c). Eliminate c_0 and c_1 from your final answer. [10]

3. **Fall 2002 Mid-Term Exam 2, Question 1 (worth 34% of the exam)**

This question addresses the formulation of wave mechanics in which the wave function is written

$$\psi(\mathbf{r}, t) = A(\mathbf{r}, t) \exp[iS(\mathbf{r}, t)/\hbar],$$

where $A(\mathbf{r}, t)$ and $S(\mathbf{r}, t)$ are real-valued functions.

We begin with a general problem described by a real Hamiltonian

$$H = P^2/2m + V(\mathbf{R}), \tag{1}$$

where P is the momentum, and V is the potential energy. Since $H^* = H$, any stationary state $\psi_\alpha(\mathbf{r})$ satisfying $H\psi_\alpha(\mathbf{r}) = E_\alpha\psi_\alpha(\mathbf{r})$ can be chosen to be real. Consider a state $\psi(\mathbf{r}, t)$ that satisfies $\psi(\mathbf{r}, 0) = \psi_\alpha(\mathbf{r}) = \psi_\alpha^*(\mathbf{r})$.

(a) [6 points] Find the functions $A(\mathbf{r}, t)$ and $S(\mathbf{r}, t)$ for the state $\psi(\mathbf{r}, t)$ defined above.

(b) [4 points] Find the "quantum potential" in the state $\psi(\mathbf{r},t)$:

$$V_Q(\mathbf{r},t) = -\frac{\hbar^2}{2m}\frac{\nabla^2 A}{A}.$$

Express your result in terms of $\psi(\mathbf{r},t)$ and symbols entering Eq. (1).

(c) [6 points] Substitute your expressions for S and V_Q into the generalized Hamilton-Jacobi equation:

$$\frac{\partial S}{\partial t} + \frac{|\boldsymbol{\nabla} S|^2}{2m} + V + V_Q = 0. \tag{2}$$

Interpret your result.

We now specialize to the case of a free particle in one dimension, i.e., $V(x) = 0$.

(d) [5 points] Find $V_Q(x,t)$ for a standing wave of wavelength h/p, corresponding to $\psi_p(x,0) = \cos(px/\hbar)$.

(e) [5 points] Find $V_Q(x,t)$ for a traveling wave of momentum p, described by a wave function $\tilde{\psi}_p(x,t)$ that satisfies the initial condition $\tilde{\psi}_p(x,0) = e^{ipx/\hbar} \neq \tilde{\psi}_p^*(x,0)$.

(f) [8 points] Equation (2) with $V_Q = 0$ describes the classical motion of a particle having momentum $\mathbf{p} = \boldsymbol{\nabla} S$. Use this fact to compare how close ψ_p and $\tilde{\psi}_p$ are to the classical limit. Explain briefly why the standing-wave and traveling-wave representations, which are related to each other by a simple unitary transformation, differ so much in their degree of "classicalness."

4. **Fall 2002 Mid-Term Exam 2, Question 2 (worth 50% of the exam)**

Consider a one-dimensional harmonic oscillator, which is subject to a strong but brief disturbance so that its Hamiltonian is

$$H = \begin{cases} \dfrac{P^2}{2m} + \dfrac{1}{2}m\omega^2 X^2 & t < 0,\ t > T, \\[2ex] \hbar\omega_1 N^2 & 0 \le t \le T, \end{cases}$$

where $N = a^\dagger a$ is the number operator.

At time $t = 0$, the state vector for this harmonic oscillator is $|\psi(t=0)\rangle = |b\rangle$, where

$$a|b\rangle = b|b\rangle, \qquad \langle b|b\rangle = 1. \tag{3}$$

Here, a is the annihilation operator and b is a real number.

(a) [12 points] Use Eqs. (3) to show that (up to an arbitrary phase)

$$|b\rangle = e^{-\frac{1}{2}b^2}\sum_{n=0}^{\infty}\frac{b^n}{\sqrt{n!}}|n\rangle,$$

where $N|n\rangle = n|n\rangle$.

(b) [4 points] The Hamiltonian for this problem satisfies $[H, \Pi] = 0$, where Π is the spatial-inversion operator (which maps x to $-x$). Does this imply that $|b\rangle$ is an eigenstate of Π?

(c) [8 points] Evaluate $\langle b|\Pi|b\rangle$.

(d) [6 points] Express $|\psi(t)\rangle$ for $0 < t < T$ in the basis $\{|n\rangle\}$.

(e) [6 points] Express $|\psi(t)\rangle$ for $t > T$ in the basis $\{|n\rangle\}$.

(f) [6 points] Simplify $|\psi(T)\rangle$ in the case $T = \pi/\omega_1$. (There should be no summations in your final answer.)

(g) [8 points] In the case $T = \pi/\omega_1$, what can you say about the product of uncertainties $\Delta X \Delta P$ at time $t > T$? (Do not explicitly calculate ΔX or ΔP.)

5. **Fall 2002 Homework 8, Question 4**

A quantum mechanical state $|\psi\rangle$ is known to be a simultaneous eigenstate of two Hermitian operators A and B which *anticommute*:

$$AB + BA = 0.$$

What can you say about the eigenvalues of A and B for state $|\psi\rangle$? Illustrate your point using the parity operator (which can be chosen to satisfy $\Pi = \Pi^{-1} = \Pi^\dagger$) and the momentum operator.

6. **Fall 2002 Homework 8, Question 5**

Ballentine Problem 13.7: Suppose that the Hamiltonian is invariant under time reversal: $[H, T] = 0$. Show that, nonetheless, an eigenvalue of T is *not* a conserved quantity.