

PHY 6645 Fall 2003 – Final Exam

Instructions: Attempt all four questions, each of which is worth 25 points. The maximum score for each part of each question is shown in square brackets. To gain full credit you should explain your reasoning and show all working. Please write neatly and remember to include your name on the front page of your answers.

Please read carefully: During this exam, you may use Shankar's *Principles of Quantum Mechanics* and lecture notes from this course. You may also use standard mathematical tables. You may quote without proof any results given in these sources; however, you should cite the source for the result (e.g., "Shankar page 25"). You are not permitted to consult any other books, notes, or papers, or to communicate with anyone other than the proctor.

In accordance with the UF Honor Code, by turning in this exam to be graded, you affirm the following pledge: *On my honor, I have neither given nor received unauthorized aid in doing this assignment.*

1. A point-like particle is described by the wave function

$$\psi(x, y, z) = A [(x + y)^2 + z^2] \exp\left(-\alpha\sqrt{x^2 + y^2 + z^2}\right),$$

where A and α are positive, real quantities.

- (a) [15 points] Express the wave function in the form $\psi = R(r)Y(\theta, \phi)$, where $Y(\theta, \phi) = \sum_{l,m} c_l^m Y_l^m(\theta, \phi)$ is a normalized angular wave-function. You should explicitly evaluate the nonzero coefficients c_l^m .
 - (b) [5 points] Give the possible outcomes of a measurement of L_z (the z component of the particle's angular momentum) and the probability of each outcome.
 - (c) [5 points] Give the possible outcomes of a measurement of L^2 (the magnitude-squared of the angular momentum) and the probability of each outcome.
2. A particle of mass μ travels in a spherical square-well potential: $V(r) = -V_0$ for $r < a$, $V(r) = 0$ for $r \geq a$, where V_0 and a are positive quantities. Consider a bound state of angular momentum $l = 0$, described by a wave function $\psi_{E00}(r, \theta, \phi) = (4\pi)^{-1/2}U_{E0}/r$ with energy $E < 0$. It will prove useful to define positive, real quantities k and κ through the relations

$$E = -\frac{\hbar^2 \kappa^2}{2\mu} = \frac{\hbar^2 k^2}{2\mu} - V_0. \quad (1)$$

- (a) [5 points] Write down the general form of a physically acceptable $U_{E0}(r)$ in the region $0 \leq r \leq a$.

Hint: Here and in (b), don't bother to use solutions of the spherical Bessel equation. Express your answer directly in terms of elementary functions.

- (b) [5 points] Write down the general form of a physically acceptable $U_{E0}(r)$ in the region $r \geq a$.
- (c) [5 points] Apply appropriate boundary conditions at $r = a$ to show that the allowed value(s) of E satisfy

$$\kappa = -k \cot ka. \quad (2)$$

- (d) [10 points] Show that there is no bound state for $V_0 < \hbar^2\pi^2/(8\mu a^2)$, and that there is at least one bound state for $V_0 > \hbar^2\pi^2/(8\mu a^2)$.

3. A beam of electrons is partially polarized by a uniform magnetic field \mathbf{B} . The electrons are in thermal equilibrium at temperature T , so their density operator is $\rho = Z^{-1} \exp(-H/k_B T)$, where H is the Hamiltonian, k_B is Boltzmann's constant and Z is a normalization factor. For the purposes of this question, ignore the orbital degrees of freedom, and take $H = a\mathbf{B} \cdot \mathbf{S} + b(\mathbf{B} \cdot \mathbf{S})^2 + c(\mathbf{B} \cdot \mathbf{S})^3$, where \mathbf{S} is the electron's spin operator, and a , b , and c are real constants.

- (a) [20 points] Express the density operator in the standard form $\rho = \frac{1}{2}(I + \mathbf{p} \cdot \boldsymbol{\sigma})$, where $\boldsymbol{\sigma}$ represents the Pauli operators, and \mathbf{p} is the polarization vector. You should be able to obtain a closed-form expression for \mathbf{p} .
- (b) [5 points] Find the ensemble expectation value $\langle \mathbf{S} \rangle$.

4. A system contains three angular momentum degrees of freedom, $j_1 = j_2 = \frac{1}{2}$, and $j_3 = 1$. Let $\mathbf{J} = \mathbf{J}_1 + \mathbf{J}_2 + \mathbf{J}_3$ be the total angular momentum of this system.

- (a) [5 points] What are the allowed eigenvalues of J_z ?
- (b) [8 points] Summarize the result of adding the three angular momenta (j_1 , j_2 , and j_3) in a shorthand notation of the form

$$j_1 \otimes j_2 \otimes j_3 = j^{(1)} \oplus \dots \oplus j^{(n)},$$

where $j^{(k)}$ represents a multiplet of total angular momentum eigenkets $\{|j^{(k)}, m\rangle\}$.

- (c) [12 points] Obtain a total angular momentum eigenstate $|j, m\rangle$ that has the largest possible value of j and the smallest possible non-negative value of m . Express your result in the product basis $|j_1, m_1\rangle \otimes |j_2, m_2\rangle \otimes |j_3, m_3\rangle$.