PHY 6645 Fall 2003 – Homework 4

Due by 5 p.m. on Friday, October 3. No credit will be available for homework submitted after 5 p.m. on Monday, October 6.

Answer questions 1–3. Please write neatly and include your name on the front page of your answers. You must also clearly identify all your collaborators on this assignment. To gain maximum credit you should explain your reasoning and show all working.

Introduction: For any piecewise-constant potential in 1D, you should be able to

- identify the range of energies E over which eigenstates of the time-independent Hamiltonian will be (i) continuously distributed in energy; (ii) forbidden on general principle; and (iii) not ruled out, but if present distributed with discrete eigenvalues.
- determine the degeneracy with which any energy E would appear, should it turn out to be an allowed eigenvalue.
- set up the form of a general solution $\psi_E(x)$ of the Schrödinger equation within each region of constant potential.
- apply the appropriate boundary conditions at each point where the potential changes.

Where relevant, you should be able (in principle at least—in practice it may be algebraically very messy) to

- formulate the equation satisfied by bound-state solutions. (It is not always straightforward to solve this equation.)
- construct (i) the wave function $\psi_{E,R}(x)$ describing a right-moving wave coming from $x = -\infty$, plus all scattered waves; and/or (ii) the wave function $\psi_{E,L}(x)$ corresponding to a left-moving wave coming from $x = +\infty$, plus all scattered waves.
- calculate the probability current density j at any point x, and calculate reflection and transmission coefficients (R and T) for waves incoming from $x = -\infty$ or $x = +\infty$.
- be able to figure out (qualitatively at least) the behavior when a wave packet approaches from either $x = -\infty$ or $x = +\infty$.

Homework questions: Explicit calculations for three standard potentials.

1. *Potential step:* $V(x) = V_0\theta(x)$ with $V_0 > 0$ (Shankar Sect. 5.4, Merzbacher Sect. 6.1)
   (a) Specify the range of E over which the state $\psi_{E,R}(x)$ exists. Construct this state explicitly for all E in this range, and calculate j, R, and T. Note: Throughout this homework set, you should specify the form of the wave function within every region of constant potential (not just within one such region).
   (b) Repeat part (a) for the state $\psi_{E,L}(x)$.
   (c) Prove that $T(E)$ must be independent of the direction of incidence for any one-dimensional system described by a real Hamiltonian. [You may assume that the system is described by a transfer matrix $P(E)$.] Verify by explicit calculation that this property is obeyed for the step potential.

(d) For situations where both $\psi_{E,L}(x)$ and $\psi_{E,R}(x)$ exist, are these two states orthogonal? If not, construct an orthogonalized pair of states to replace them.

2. *Rectangular barrier:* $V(x) = V_0\theta(a - |x|)$ with $V_0, a > 0$ (Merzbacher Sect. 6.2)
   (a) Specify the range of E over which the state $\psi_{E,R}(x)$ exists. Construct this state explicitly for all E in this range, and calculate j, R, and T.
   (b) Construct the transfer matrix $P(E)$ for this potential by suitably combining the transfer matrices for two step potentials. Verify that your result agrees with that obtained in part (a).
   (c) Identify the energy of any resonances, i.e., peaks in $T(E)$.
   (d) Construct the energy eigenstates for two special cases: (i) the infinite potential barrier, $V_0 = \infty$; (ii) the delta-function barrier, $V(x) = 2V_0 a\delta(x)$.

3. *Rectangular well:* $V(x) = -V_0\theta(a - |x|)$ with $V_0, a > 0$ (Shankar Sect. 5.2, Merzbacher Sect. 6.4)
   (a) Specify the range of E over which the state $\psi_{E,R}(x)$ exists. Construct this state explicitly for all E in this range, and calculate j, R, and T.
   (b) Construct the transfer matrix $P(E)$ for this potential by suitably combining the transfer matrices for two step potentials. Verify that your result agrees with that obtained in part (a).
   (c) Identify the energy of any resonances in the range where $\psi_{E,R}(x)$ exists.
   (d) Derive the equation that determines the bound-state energies.
   (e) Find all energy eigenvalues and eigenstates for the special case of a delta-function well, $V(x) = -2V_0 a\delta(x)$ (c.f. Shankar Exercise 5.2.3).
   (f) Do Shankar Ex. 5.2.2, proving that every attractive potential in one dimension has at least one bound state. (This need not be the case in higher dimensions.)

Additional exercises: For practice only—not to be turned in for grading.

- Check that you understand what happens when a Gaussian wave packet scatters from the step potential. Shankar has a good discussion of this.

- Work through Shankar Exercise 5.2.6, dealing with the graphical procedure for finding the bound-state energies of the general rectangular well.

- For the rectangular barrier and the rectangular well, understand the qualitative effects of perturbing the potential in various ways, e.g., making $V(x) = V_1$ for $x > a$, where $0 < V_1 \ll V_0$, or making $V(x)$ vary linearly from V_0 at $x = -a$ to V_1 at $x = a$.

- *Double Barriers:* $V(x) = V_0\theta(|x| - a)\theta(b - |x|)$ with $V_0 > 0$ and $b > a > 0$.
  1. Construct the transfer matrix element for this potential by combining the transfer matrices for two rectangular barriers.
  2. Plot the transmission coefficient T vs E/V_0 for $a = \sqrt{18\hbar^2/mV_0}$ and (i) $b = 1.2a$; (ii) $b = 3a$; (iii) $b = 10a$. It will suffice to calculate $T(E)$ numerically.

  This potential is somewhat more complicated than those above. It is included as a challenge for anyone who is already familiar with the more standard cases.