PHY 6645 Fall 2003 – Homework 5

Due by 5 p.m. on Friday, October 10. No credit will be available for homework submitted after this deadline. Solutions to this assignment will be made available outside Kevin Ingersent's office immediately after the deadline.

Answer all parts of the question. Please write neatly and include your name on the front page of your answers. You must also clearly identify all your collaborators on this assignment. To gain maximum credit you should explain your reasoning and show all working.

A quantum mechanical system can be represented by a three-dimensional vector space spanned by an orthonormal basis $\{|j\rangle,\ j = 1,\ 2,\ 3\}$.

We will initially describe this system in the *Schrödinger picture*, in which the Hamiltonian for the system can be written in the outer-product form

$$H = 2\epsilon|3\rangle\langle 3| - \epsilon|1\rangle\langle 1| + \eta(|2\rangle\langle 3| + |3\rangle\langle 2|),$$

where ϵ and η are real scalars that satisfy $0 < \eta < \epsilon$. We will also consider another Hermitian operator Λ, which can be written

$$\Lambda = \lambda(|1\rangle\langle 1| - |2\rangle\langle 2|) + 2\lambda|3\rangle\langle 3|.$$

(a) Find the matrix representations of H and Λ in the basis $\{|j\rangle\}$.

(b) Find the eigenvalues E_j, and the normalized eigenkets $|E_j\rangle$, such that $H|E_j\rangle = E_j|E_j\rangle$, $j = 1,\ 2,\ 3$. Label your eigensolutions so that $E_1 \le E_2 \le E_3$. Express each eigenket in the basis $\{|j\rangle\}$, and choose the overall phase of the eigenket so that $\langle j|E_j\rangle$ is real and positive.

(c) Find the matrix representations of H, Λ, and the propagator $U(t)$ in the basis $\{|E_j\rangle\}$ of energy eigenkets.

(d) Write down the spectral decompositions of H, Λ, and $U(t)$, i.e., write each operator as a weighted sum of projection operators.

(e) Suppose that at time $t = 0$, the state vector is $(|1\rangle - |2\rangle)/\sqrt{2}$. Express the state vector $|\psi(t)\rangle$ for general t in the basis $\{|j\rangle\}$. Calculate the expectation values of H and Λ at time t.

(f) Suppose instead that at time $t = 0$, the state vector is $|E_2\rangle$. Express the state vector $|\psi(t)\rangle$ in the basis $\{|j\rangle\}$. Calculate the expectation values of H and Λ at time t.

Now let us switch to the *Heisenberg picture*, in which we label the state vector $|\psi\rangle_H$ and the Hamiltonian H_H.

(g) Find the matrix representations of $H_H(t)$ and $\Lambda_H(t)$ in the basis $\{|E_j\rangle\}$.

(h) Find the matrix representations of $H_H(t)$ and $\Lambda_H(t)$ in the basis $\{|j\rangle\}$.

(i) Consider the initial condition described in (e). Express the state vector $|\psi(t)\rangle_H$ in the basis $\{|j\rangle\}$. Calculate the expectation values of H and Λ at time t, and verify that the results are the same as in (e).

(j) Consider the initial condition described in (f). Express the state vector $|\psi(t)\rangle_H$ in the basis $\{|j\rangle\}$. Calculate the expectation values of H and Λ at time t, and verify that the results are the same as in (f).

Finally, let us switch to the *interaction picture*, based on a decomposition of the Schrödinger-picture Hamiltonian into an unperturbed part

$$H^{(0)} = 2\epsilon|3\rangle\langle 3| - \epsilon|1\rangle\langle 1|$$

and an interaction term

$$H^{(I)} = \eta(|2\rangle\langle 3| + |3\rangle\langle 2|).$$

In this picture, we label the state vector $|\psi\rangle_I$ and the Hamiltonian H_I.

(k) Find the matrix representation of $H_I(t)$ and $\Lambda_I(t)$ in the basis $\{|E_j\rangle\}$.

(l) Find the matrix representation of $H_I(t)$ and $\Lambda_I(t)$ in the basis $\{|j\rangle\}$.

(m) Consider the initial condition described in (e). Express the state vector $|\psi(t)\rangle_I$ in the basis $\{|j\rangle\}$. Calculate the expectation values of H and Λ at time t, and verify that the results are the same as in (e).

(n) Consider the initial condition described in (f). Express the state vector $|\psi(t)\rangle_I$ in the basis $\{|j\rangle\}$. Calculate the expectation values of H and Λ at time t, and verify that the results are the same as in (f).

Note that in all three pictures, the basis vectors $|j\rangle$ and $|E_j\rangle$ are independent of time.