

PHY 6645 Fall 2003 – Homework 7

Due by 5 p.m. on Friday, October 31. No credit will be available for homework submitted after 5 p.m. on Monday, November 3.

Answer both questions. Please write neatly and include your name on the front page of your answers. You must also clearly identify all your collaborators on this assignment. To gain maximum credit you should explain your reasoning and show all working.

1. *The limit of large quantum numbers.* For each of the eigenstates n of the harmonic oscillator listed below under (a)–(d), generate a single graph plotting the following four quantities (clearly labeled):

- (i) The quantum-mechanical probability density $P_{\text{QM}}(y) = |\psi_n(y)|^2$ at dimensionless coordinate $y = x\sqrt{m\omega/\hbar}$;
- (ii) The classical probability density $P_{\text{CM}}(y) \propto 1/v(y)$, where $v(y)$ is the speed of a classical particle having the same energy as the n 'th quantum-mechanical eigenstate when that particle is located at coordinate y ;
- (iii) The box-averaged quantum-mechanical probability density

$$\bar{P}_{\text{QM}}(y, w) = \frac{1}{w} \int_{y-w/2}^{y+w/2} P_{\text{QM}}(y) dy. \quad (1)$$

Use the value of w specified below.

- (iv) The box-averaged classical probability density $\bar{P}_{\text{CM}}(y, w)$, defined by analogy with $\bar{P}_{\text{QM}}(y, w)$. Note that $\bar{P}_{\text{CM}}(y, w)$ may be nonzero even if y lies outside the classical turning points, so long as $y - w/2$ or $y + w/2$ is classically accessible.

The horizontal axis should extend over the range $0 \leq y \leq y_{\text{max}}$, where y_{max} is specified below. Use the same vertical scale for all four probability densities, choosing this scale so that the graph contains all of curves (i), (iii), and (iv). [$P_{\text{CM}}(y)$ diverges at certain points, so curve (ii) cannot be entirely contained.] Show the scale on each axis.

Generate such a graph for

- (a) $n = 0$, $w = 0.75$, $y_{\text{max}} = 4$;
- (b) $n = 1$, $w = 0.75$, $y_{\text{max}} = 4$;
- (c) $n = 14$, $w = 0.75$, $y_{\text{max}} = 8$;
- (d) $n = 15$, $w = 0.75$, $y_{\text{max}} = 8$.

To answer this question, you will need to produce some sort of computer program. You may use any general-purpose programming language (e.g., Fortran or C) or a higher-level programming environment (such as Mathematica, Matlab, or Maple). You may choose to represent the wave function as a set of numerical values calculated at a sufficiently fine grid of y points, or as an algebraic function of y which you can manipulate symbolically. The main restrictions are that you should program your own solution to the problem, not merely copy someone else's—on this question, collaboration should be limited to discussing computational methods and comparing end results. You should not use any predefined software for generating the harmonic oscillator wave functions $\psi_n(y)$, but it is acceptable to use standard library functions for the Hermite polynomials.

2. *Coherent states.* A coherent state of the harmonic oscillator is a state of the form $|z\rangle = \exp(za^\dagger - \frac{1}{2}|z|^2)|0\rangle$, where $|0\rangle$ is the ground state, a^\dagger is the creation operator, and z is any complex number. Coherent states have many remarkable characteristics that make them useful in quantum optics and in the path-integral formulation of quantum mechanics. The purpose of this question is to acquaint you with some of their basic properties. The answers to several parts of the question can be gleaned from discussions of coherent states in Shankar (pages 607–613) and Merzbacher (Section 10.7).

- (a) By expanding $\exp(za^\dagger - \frac{1}{2}|z|^2)$, express $|z\rangle$ as a sum over all number eigenstates $|n\rangle$ of the harmonic oscillator. Use this expression for $|z\rangle$ to calculate $\langle z_1|z_2\rangle$, and confirm that $|z\rangle$ is normalized to unity.
- (b) Calculate $\langle z_1|a|z_2\rangle$ and $\langle z_1|a^\dagger|z_2\rangle$. Use your results to calculate $\langle z_1|X|z_2\rangle$ and $\langle z_1|P|z_2\rangle$.
- (c) Demonstrate that $|z\rangle$ is an eigenket of the annihilation operator a with eigenvalue z . Use this fact to derive and solve a differential equation for the wave function $\psi_z(x) = \langle x|z\rangle$.
- (d) Suppose that a harmonic oscillator is initially in state $|\psi(0)\rangle = |z\rangle$. Using the representation of $|z\rangle$ derived in part (a), find $|\psi(t)\rangle$. Simplify your answer as much as possible to show that $|\psi(t)\rangle \propto |ze^{-i\omega t}\rangle$, and find the constant of proportionality.
- (e) For the state $|\psi(t)\rangle$ of part (d), find $\langle E\rangle$, ΔE , $\langle X\rangle$, ΔX , $\langle P\rangle$, and ΔP at time t . Use the number basis to obtain your answers.
- (f) Use the preceding results to show that there exists an infinite set of Gaussian wave functions, each of which (i) is a minimum-uncertainty state (with respect to $\Delta X\Delta P$), and (ii) remains a minimum-uncertainty state at all times t when allowed to propagate in a harmonic potential.
- (g) Which coherent states (if any) are stationary states?
- (h) Is a linear combination of two different coherent states $|z_1\rangle$ and $|z_2\rangle$ (where $z_1 \neq z_2$) ever a coherent state?
- (i) Explain why the creation operator a^\dagger has no eigenket besides the null vector. (Important note: The harmonic-oscillator ground state conventionally denoted $|0\rangle$ is **not** the null vector for the Hilbert space.)