PHY 6645 Fall 2003 – Homework 8

Due by 5 p.m. on Monday, November 10. No credit will be available for homework submitted after 5 p.m. on Wednesday, November 12.

Answer all questions. Please write neatly and include your name on the front page of your answers. You must also clearly identify all your collaborators on this assignment. To gain maximum credit you should explain your reasoning and show all working.

1. Consider a particle confined to a one-dimensional box of width L. Calculate $\Delta X\,\Delta P$ in the n'th energy eigenstate ($n = 1$ being the ground state). Verify that $\Delta X\,\Delta P \geq \hbar/2$ for every energy eigenstate.

2. A spin-$\frac{1}{2}$ degree of freedom can be described by a two-dimensional vector space, spanned by orthonormal basis vectors $|{\uparrow}\rangle$ (spin-z "up") and $|{\downarrow}\rangle$ (spin-z "down"). The operators describing the Cartesian components of the spin are

$$S_x = \frac{\hbar}{2}\left(|{\downarrow}\rangle\langle{\uparrow}| + |{\uparrow}\rangle\langle{\downarrow}|\right), \quad S_y = i\,\frac{\hbar}{2}\left(|{\downarrow}\rangle\langle{\uparrow}| - |{\uparrow}\rangle\langle{\downarrow}|\right), \quad S_z = \frac{\hbar}{2}\left(|{\uparrow}\rangle\langle{\uparrow}| - |{\downarrow}\rangle\langle{\downarrow}|\right).$$

   (a) Verify be explicit calculation that the spin operators satisfy the commutation relations $[S_x, S_y] = i\hbar S_z$. Use this fact and the general inequality $\Delta\Omega\,\Delta\Lambda \geq \frac{1}{2}|\langle[\Omega, \Lambda]\rangle|$ to derive an uncertainty relation connecting S_x and S_y.

   (b) Calculate ΔS_x and ΔS_y for an arbitrary normalized ket written in the form

$$|\psi\rangle = e^{i\xi}\left(\cos\theta\,|{\uparrow}\rangle + e^{i\phi}\sin\theta\,|{\downarrow}\rangle\right),$$

   where ξ, θ, and ϕ are real quantities.

   (c) Find a non-null ket that minimizes the product $\Delta S_x\,\Delta S_y$. Verify that this ket satisfies the appropriate uncertainty relation.

   (d) Find two, physically inequivalent kets that maximize the product $\Delta S_x\,\Delta S_y$.

3. Consider two Hermitian operators Ω and Λ that contain no explicit time dependence in the Schrödinger picture. Let $[\Omega, \Lambda] = i\Gamma$. Suppose that in a particular system Ω and Λ are both constants of motion.

   (a) Show that Γ is also a constant of motion for this system.

   (b) Show that if the eigenspectrum of H for this system contains no degeneracies, then Γ must be identically zero. Hint: As part of your answer, evaluate $\Omega|E\rangle$, $\Lambda|E\rangle$, and hence $\Gamma|E\rangle$, for an arbitrarily chosen energy eigenket $|E\rangle$.

   (c) Conversely, show that if $\Gamma \neq 0$, there must be at least one degenerate eigenvalue. Hint: Argue that there must be at least one energy eigenket $|E\rangle$ such that $\Gamma|E\rangle \neq 0$. Prove that this energy eigenket must be degenerate.

4. Consider a one-dimensional system that has a time-independent Hamiltonian.

   (a) What are the physically acceptable eigenvalues and eigenkets of the time displacement operator $U(s)$?

   (b) What are the physically acceptable eigenvalues and eigenkets of the spatial displacement operator $S(a)$?