Rotation Through 2π: SO(3) vs SU(2), Superselection, and Time Reversal

- It is a general feature of all states of half-integer angular momentum that any rotation through an angle of 2π,

$$U[R(2\pi\hat{\boldsymbol{\omega}})] \equiv U[R(2\pi)] = -1.$$

  - We saw this explicitly for $j = \frac{1}{2}$ (actually, $s = \frac{1}{2}$).
  - The proof is also straightforward for arbitrary j in the case $\hat{\boldsymbol{\omega}} = \hat{\mathbf{z}}$:

$$\begin{aligned} U[R(2\pi\hat{\mathbf{z}})]\,|j,m\rangle &= \exp(-i2\pi J_z/\hbar)\,|j,m\rangle \\ &= \exp(-i2\pi m)\,|j,m\rangle \\ &= (-1)^{2j}|j,m\rangle, \end{aligned}$$

    since $m = j - k$, where k is an integer.
  - For arbitrary j and $\hat{\boldsymbol{\omega}}$, it is necessary to rotate the $|j,m\rangle$'s into eigenstates of $\hat{\boldsymbol{\omega}}\cdot\mathbf{J}$, apply $U[R(2\pi\hat{\boldsymbol{\omega}})]$, then rotate back. The conclusion is again

$$U[R(2\pi\hat{\mathbf{z}})]\,|j,m\rangle = (-1)^{2j}|j,m\rangle.$$

  - Important: $U[R(2\pi)] = -1$ means that the state vector is multiplied by -1, *not* that the spin (or its expectation value $\langle\mathbf{S}\rangle$) changes sign.

- This peculiar feature can be traced to the properties of the matrices that represent the symmetry operators:
  - Spatial rotations have the SO(3) group properties of 3×3 special orthogonal matrices, for which a 2π rotation equals the identity.
  - The rotation operators for a spin-$\frac{1}{2}$ system, $U[R(\boldsymbol{\omega})] = \cos(\omega/2)\,I - i\sin(\omega/2)\,\hat{\boldsymbol{\omega}}\cdot\boldsymbol{\sigma}$, span the set of 2×2 unitary unimodular[1] matrices

$$U = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix},$$

    where $|a|^2 + |b|^2 = 1$. This set forms the group SU(2) under matrix multiplication.
  - There is a 2:1 mapping of the elements of SU(2) onto those of SO(3). Formally, $U(a,b)$ and $U(-a,-b)$ correspond to the same 3×3 matrix of SO(3).
  - Strictly, the $D^{(j)}$ matrices introduced previously are irreducible representations of SU(2). The odd-dimensional (integer j) representations do not preserve the distinction between ω and $\omega + 2\pi$ rotations; these matrices also serve as representations of SO(3).

[1] A general 2×2 unitary matrix can be written $U(a,b) = \begin{pmatrix} a & b \\ -b^* e^{i\theta} & a^* e^{i\theta} \end{pmatrix}$, where $|a|^2 + |b|^2 = 1$ and θ is real; then $\det U = e^{i\theta}$. "Unimodular" means $\det U = 1$, or $\theta = 2\pi$ times an integer.

- We have seen that that rotations through ω and $\omega+2\pi$ about the same axis are identical for spatial rotations, but inequivalent for spin rotations.

  However, we will take it as axiomatic that all physical observables Ω are invariant under 2π rotations, i.e.,

$$[\Omega,\, U[R(2\pi)]\,] = 0.$$

  This assumption leads to a *superselection rule:* no operator corresponding to a physical observable can have nonvanishing matrix elements between a state $|+\rangle$ of integer angular momentum and a state $|-\rangle$ of half-integer angular momentum.

  Proof:

$$\begin{aligned} \langle +|\, U[R(2\pi)]\,\Omega\,|-\rangle &= \langle +|\,\Omega\, U[R(2\pi)]\,|-\rangle, \\ +\langle +|\Omega|-\rangle &= -\langle +|\Omega|-\rangle, \\ \langle +|\Omega|-\rangle &= 0. \end{aligned}$$

- Finally, 2π rotations are connected with *time-reversal symmetry:*

  Recall that in the passive picture, we defined the time-reversal operator T by

$$\begin{aligned} \mathbf{R}' &= T^{-1}\,\mathbf{R}\,T = \mathbf{R}, \\ \text{and}\quad \mathbf{P}' &= T^{-1}\,\mathbf{P}\,T = -\mathbf{P}, \\ \Rightarrow\quad \mathbf{L}' &= T^{-1}\,\mathbf{L}\,T = -\mathbf{L}. \end{aligned}$$

  In order to be consistent, we must require

$$\mathbf{S}' = T^{-1}\,\mathbf{S}\,T = -\mathbf{S}.$$

  As shown by Ballentine (p. 382), the form of the time-reversal operator appropriate for the representation $\psi_{\rm o}(\mathbf{r},t)\chi(t) = \langle \mathbf{r}| \otimes \langle s,m|\psi(t)\rangle$ is

$$T = \exp(-i\pi S_y/\hbar)\, C,$$

  where C is the complex conjugation operator (acting on both $\psi_{\rm o}$ and χ).

  Recalling that $iS_y = \frac{1}{2}(S_+ + S_-)$, where $S_\pm|s,m\rangle = \sqrt{(s\mp m)(s\pm m+1)}\hbar|s,m\pm 1\rangle$, we see that $\exp(-i\pi S_y/\hbar)$ has a *real* matrix representation in the basis of common eigenkets of S^2 and S_z. Then

$$\begin{aligned} T^2 &= \exp(-i\pi S_y/\hbar)\, C\, \exp(-i\pi S_y/\hbar)\, C \\ &= \exp(-i\pi S_y/\hbar)\, \exp(-i\pi S_y/\hbar)\, C\, C \\ &= \exp(-i2\pi S_y/\hbar). \end{aligned}$$

  Since $\exp(-i2\pi L_y/\hbar) = 1$ always, and $[S_y, L_y] = 0$, we can write

$$T^2 = \exp(-i2\pi J_y/\hbar) \equiv U[R(2\pi)] = (-1)^{2j}.$$

  This justifies the claim made previously that $T^2 = +1$ [-1] for particles having integer [half-integer] angular momentum.