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The WKB Connection Formulae

The connection formulae, used to relate the WKB wave functions on either side of a classical turning point (located at $x = a$), can be summarized as follows:

$$\psi_-(x) \quad \longrightarrow \quad \sqrt{\pi|l|}Ai\left(\frac{x-a}{l}\right) \quad \longrightarrow \quad 2\psi_c(x) \tag{1}$$

$$\psi_+(x) \quad \longleftarrow \quad -\sqrt{\pi|l|}Bi\left(\frac{x-a}{l}\right) \quad \longleftarrow \quad -\psi_s(x) \tag{2}$$

where

$$\psi_\pm(x) = \frac{1}{\sqrt{\kappa(x)}} \exp\left[\pm\int \kappa(x')dx'\right], \qquad l = \left(\frac{\hbar^2}{2m|g|}\right)^{1/3} \operatorname{sgn} g, \quad g = \left.\frac{dV}{dx}\right|_{x=a}, \tag{3}$$

$$\psi_c(x) = \frac{1}{\sqrt{k(x)}} \cos\left[\int k(x')dx' - \frac{\pi}{4}\right], \qquad \psi_s(x) = \frac{1}{\sqrt{k(x)}} \sin\left[\int k(x')dx' - \frac{\pi}{4}\right]. \tag{4}$$

All integration is carried out from $\min(x, a)$ to $\max(x, a)$, so each integral has a non-negative value which grows with $|x - a|$. With this convention, Eqs. (1)–(4) apply irrespective of the sign of g.

The connection formulae are exact only in the limit $\epsilon \to 0^+$, where $\epsilon = \sqrt{\hbar^2/(2ml_0^2V_0)}$ is the small parameter entering the WKB treatment of the potential $V(x) = V_0 w(x/l_0)$.

For $\epsilon > 0$, errors arising from use of the connection formulae will be minimized if Eqs. (1) and (2) are applied in the direction of the arrows:

1. If the wave function is proportional to ψ_c in the classically allowed region, one *cannot* deduce that the wave function on the other side of the turning point is strictly proportional to ψ_- — only that the coefficient of ψ_+ is subleading in ϵ. Neglect of a ψ_+ component with even a very small coefficient would have severe consequences, because this component grows exponentially away from the turning point, and at sufficiently large distances must overshadow the exponentially shrinking ψ_- component.

   However, if $V(x) > E$ for all x on one side of the turning point, say $x > a$, the requirement that $\psi(x) \to 0$ for $x \to \infty$ ensures that coefficient of ψ_+ is identically zero. Then the WKB solution for $x < a$ is well-predicted by Eq. (1). The effect of finite ϵ is at worst to introduce an error in the phase of the oscillatory solution.

2. Equation (2) need only be utilized in problems involving tunneling through a finite-width barrier, inside which the WKB wave function will generally have nonzero coefficients of both ψ_+ and ψ_-. If we use Eq. (2) in the reverse direction, then in the classically allowed region we neglect a subleading ψ_s component; this omission can lead to a large error in the phase of the oscillatory wave function. Application of Eq. (2) in the direction shown results in neglect of a subleading ψ_- component in the forbidden region. The consequences of this omission are minimal, since this ψ_- component shrinks at an exponential rate away from the turning point.