

## PHY 6646 Spring 2002 – Homework 2

Due by 5 p.m. on Friday, January 25. No credit will be available for solutions submitted after 5 p.m. on Monday, January 28.

Answer all questions. To gain full credit you should explain your reasoning and show all working. Please write neatly and include your name on the front page of your answers.

1. Consider an isolated point-like particle of mass  $m$  and charge  $q$ , situated in a uniform magnetic field  $\mathbf{B}$ , and described by the Hamiltonian

$$H = \frac{1}{2m} \left| \mathbf{P} - \frac{q}{c} \mathbf{A} \right|^2 + q\phi. \quad (1)$$

Let us consider the family of gauges  $\phi(\mathbf{r}) = 0$ ,  $\mathbf{A}^{(\Lambda)}(\mathbf{r}) = \frac{1}{2} \mathbf{B} \times \mathbf{r} - \nabla \Lambda(\mathbf{r})$ , where each gauge is specified by a different function  $\Lambda(\mathbf{r})$ .

The real-space wave function corresponding to an abstract state  $|\psi\rangle$  is gauge-dependent; let us denote it  $\psi^{(\Lambda)}(\mathbf{r})$ .

The particle's orbital magnetic moment operator  $\boldsymbol{\mu}_L$  can be defined (by analogy with classical electromagnetism) to be the operator which enters the Hamiltonian in the combination  $-\mathbf{B} \cdot \boldsymbol{\mu}_L$ . The explicit form of this operator is also gauge-dependent, so we denote it  $\boldsymbol{\mu}_L^{(\Lambda)}$ .

- (a) Substitute the coordinate-representations of  $\mathbf{P}$ ,  $\phi$  and  $\mathbf{A}^{(\Lambda)}$  into the Hamiltonian (1), expand all parentheses, and then simplify the resulting expression.
- (b) Find an explicit expression for  $\boldsymbol{\mu}_L^{(\Lambda)}$ . Make it resemble as closely as possible its form in the gauge  $\Lambda = 0$ , i.e.,  $\boldsymbol{\mu}_L^{(0)} = q/(2mc)\mathbf{L}$ , where  $\mathbf{L}$  is the orbital angular momentum.
- (c) Show that the expectation value of  $\boldsymbol{\mu}_L^{(\Lambda)}$  in the state  $\psi^{(\Lambda)}(\mathbf{r})$  is gauge-independent.
- (d) Show that if the magnetic operator is given the alternative definition  $\boldsymbol{\mu}_L^{(\Lambda)} = -\partial H^{(\Lambda)}/\partial \mathbf{B}$ , then this operator can be expressed in the gauge-independent form

$$\boldsymbol{\mu}_L = \frac{q}{2c} \mathbf{R} \times \mathbf{V},$$

where  $\mathbf{V}$  is the velocity operator.

2. Based on Ballentine Problem 12.2: The most general state operator for a spin- $\frac{1}{2}$  system has the form  $\rho = \frac{1}{2}(I + \mathbf{p} \cdot \boldsymbol{\sigma})$ , where  $\sigma_j$  ( $j = 1, 2, 3$ ) is a Pauli operator and  $\mathbf{p}$  is the (real) polarization vector, whose length cannot exceed 1. If the system has a magnetic moment  $\boldsymbol{\mu} = \frac{1}{2}\gamma\hbar\boldsymbol{\sigma}$  and is in a constant, uniform magnetic field  $\mathbf{B}$ , calculate the time dependence of  $\rho(t)$  in the Schrödinger picture. Describe the result geometrically in terms of the variation of the vector  $\mathbf{p}$ .
3. Shankar Exercise 14.4.3.

Before tackling this question, you may find it helpful to read the section “Paramagnetic Resonance” on pp. 392–394 of Shankar.