

PHY 6646 Spring 2002 – Homework 4

Due by 5 p.m. on Friday, February 8. No credit will be available for solutions submitted after 5 p.m. on Tuesday, February 12.

Answer all questions. To gain full credit you should explain your reasoning and show all working. Please write neatly and include your name on the front page of your answers.

1. Consider the one-dimensional power-law potential

$$V(x) = V_0|x/a|^\alpha,$$

where V_0 , a , and α are positive constants.

- (a) Use the WKB method to approximate the allowed energies for general α .
- (b) Compare the WKB value for the ground state energy with either the exact value or a variational estimate for the each of the following cases: (i) $\alpha = 1$; (ii) $\alpha = 2$; (iii) $\alpha = 4$.
- (c) Show that the connection formulae are valid only for stationary states whose energies exceed a threshold value $E_{\min}(\alpha)$. Obtain an expression for $E_{\min}(\alpha)$, and determine whether or not the connection formulae are valid for each of the ground-state wave functions considered in (b).
- (d) Estimate the number of bound states of the truncated power-law potential

$$V(x) = \begin{cases} V_0|x/a|^\alpha & \text{for } |x| \leq a, \\ V_0 & \text{for } |x| \geq a. \end{cases}$$

Note that WKB method predicts that there is no bound state unless V_0 exceeds a nonzero minimum value. This is an artifact of the approximations inherent in the method, because there is a general theorem (see Shankar Exercise 5.5.5) that every attractive potential in one dimension has at least one bound state.

2. Consider the application of the WKB method to the tunneling of particles through a potential barrier described by a smoothly varying one-dimensional potential $V(x)$. Suppose that the WKB wave functions in the classically accessible regions to the left ($j = L$) and right ($j = R$) of the barrier are written in the form

$$\psi_j(x) = \frac{A_j}{\sqrt{k(x)}} \exp \left[i \int_{a_j}^x k(x') dx' \right] + \frac{B_j}{\sqrt{k(x)}} \exp \left[-i \int_{a_j}^x k(x') dx' \right],$$

where a_L [a_R] is the coordinate of the classical turning point on the left [right] side of the barrier.

Prove the formula given in class:

$$\begin{pmatrix} A_R \\ B_R \end{pmatrix} = \begin{pmatrix} \cosh \alpha & -i \sinh \alpha \\ i \sinh \alpha & \cosh \alpha \end{pmatrix} \begin{pmatrix} A_L \\ B_L \end{pmatrix},$$

where

$$\alpha = \int_{a_L}^{a_R} \kappa(x) dx + \ln 2.$$

3. Use the WKB method to treat a potential barrier described by

$$V(x) = \begin{cases} V_0 [1 - (x/a)^2] & \text{for } |x| \leq a, \\ 0 & \text{for } |x| \geq a, \end{cases}$$

where V_0 and a are positive constants.

- (a) Find the transmission coefficient for an incident plane wave of energy $0 < E < V_0$.
- (b) Find the transmission coefficient for an incident plane wave of energy $E > V_0$.
- (c) Sketch the variation of the transmission coefficient as a function of E over the range $0 < E < 3V_0$.

In answering the problems above, you may need to evaluate some unfamiliar integrals. A good resource is I. S. Gradshteyn and I. M. Ryzhik, *Tables of Integrals, Series, and Products*, six editions (Academic Press).