

PHY 6646 Spring 2002 – Homework 5

Due by 5 p.m. on Friday, February 15. On-time solutions will be returned in class on Wednesday, February 20 (ahead of the mid-term to be held on February 22). No credit will be available for solutions submitted after 5 p.m. on Monday, February 18.

Answer all questions. To gain full credit you should explain your reasoning and show all working. Please write neatly and include your name on the front page of your answers.

1. This problem is quite long and involved. It involves a more sophisticated treatment of the problem of α decay than appears in Shankar, or in most other texts. However, by the time you have worked through it, you will have the satisfaction of having reproduced a famous application of the WKB method from nuclear physics.

The decay of a nucleus of charge $(Z+2)e$ into a nucleus of charge Ze plus an α particle can be crudely modeled as the problem of an α particle in a spherically symmetric potential

$$V(r) = \begin{cases} -V_0 & \text{for } r < r_0, \\ 2Ze^2/r & \text{for } r > r_0, \end{cases}$$

where r_0 is roughly the nuclear radius, and $V_0 > 0$. The α decay corresponds to the tunneling of an α particle of energy $E > 0$ from the region $r < r_0$ into the region $r > a(E)$, where $a(E)$ is the classical turning point for an α particle of energy E approaching from $r = \infty$. The physical limit is one in which $E, V_0 \ll V(r_0) = 2Ze^2/r_0$, and $a \gg r_0$. For simplicity, we will assume that the escaping α particle is always in an s -wave state.

- (a) Set up the radial Schrödinger wave equation for $u(r) = \sqrt{4\pi r}\psi(r)$.
- (b) Write down the general form of the WKB wave function for an s -wave state of energy $E > 0$. You will need to provide a separate expression for $u(r)$ in each of the ranges $0 < r < r_0$, $r_0 < r < a$, and $r > a$.
- (c) Apply the four boundary conditions that must be obeyed by the wave function:
 - i. The exact boundary condition at $r = 0$.
 - ii. The requirement that the solution becomes a pure-outgoing wave for $r \rightarrow \infty$.
 - iii. The connection formulae at $r = a$.
 - iv. Continuity of u and du/dr at $r = r_0$. Application of this boundary condition requires some care. To simplify things, assume that the barrier is sufficiently broad and high that you can neglect the component of the wave function that grows exponentially as r increases into the classically forbidden region. Also, in the spirit of the WKB approximation, when evaluating du/dr for $r > r_0$, you should neglect the term arising from the derivative of the amplitude of u (which amounts to neglecting $d\kappa/dx$ compared to κ^2).

Hence express the wave function throughout all space in terms of the amplitude A_0 of the wave function in the region $r < r_0$.

- (d) Calculate the probability P of finding the α particle inside the nucleus (i.e., in the region $r < r_0$). This allows you to eliminate A_0 from the wave function and to express $u(r)$ everywhere as a multiple of P .

- (e) You should find that the amplitude of the wave function in the region $r > a$ depends on the integral

$$C = \hbar^{-1} \int_{r_0}^a dr \sqrt{2m[V(x) - E]}$$

Show that

$$C = \frac{Ze^2}{\hbar} \sqrt{\frac{2m}{E}} (2\phi - \sin 2\phi)$$

$$\xrightarrow{a \gg r_0} c_1 Z / \sqrt{E} - c_2 \sqrt{Zr_0},$$

where $\cos \phi = \sqrt{r_0 E / (2Ze^2)}$, and c_1 and c_2 are independent of Z , r_0 , and E .

- (f) Calculate the probability current I describing the rate at which particles move outward through a sphere of radius $r > a$ centered on the origin. The probability current through any surface is $I = \int \mathbf{j} \cdot d\mathbf{S}$, where

$$\mathbf{j} = \frac{-i\hbar}{2m} (\psi^* \nabla \psi - \psi \nabla \psi^*)$$

is the standard probability current density.

- (g) Equating I from part (f) with $-dP/dt$, find an expression for $P(t)$, the probability that a given nucleus survives at least a time t without decaying. From this determine the half-life $\tau_{1/2}$ defined by $P(\tau_{1/2}) = \frac{1}{2}P(0)$.
- (h) Show that the dominant E and Z dependence of the half-life is a linear variation of $\ln \tau_{1/2}$ with C defined in (e). This relation is observed experimentally for many alpha emitters. [C can be estimated under two assumptions: (i) $r_0 \propto A^{1/3}$, where A is the atomic mass; (ii) $E = \Delta mc^2$, where Δm is the difference between the rest mass of the original nucleus and the combined rest mass of the product nucleus and the emitted α particle.]

2. Shankar Exercise 17.2.1, with the following addition:

(1.5) Recall that the first-order correction to the unperturbed ground-state energy always provides an upper bound on the true ground-state energy. Can you lower this variational bound using a family of trial wave functions that includes the unperturbed ground state?

3. Shankar Exercise 17.2.2.

Treat just the spin degrees of freedom in this problem (i.e., ignore the orbital degrees of freedom).