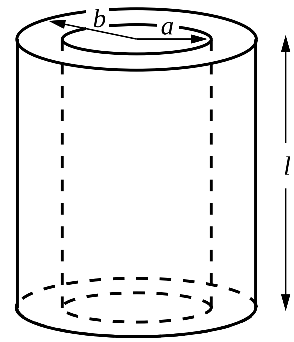
PHY 6646 Spring 2003 K. Ingersent

The Bound-State Aharonov-Bohm Effect

There are a number of situations in which the electromagnetic vector potential affects the physical properties of a particle, even though the electric and magnetic potentials are identically zero throughout the region accessible to the particle. The best-known example is the Aharonov-Bohm effect (Shankar pp. 497–499).



However, a variant—the "bound-state Aharonov-Bohm effect" (Ballentine pp. 323–325)—is easier to analyze rigorously in the wave-function formulation of quantum mechanics:

A particle of mass M and charge q is confined to the interior of a toroidal box, so that its cylindrical coordinates (ρ, ϕ, z) satisfy $a < \rho < b$, $0 < z < l$. The magnetic field is nonzero in the region $\rho < a$, but vanishes everywhere inside the box.

We take the electromagnetic scalar potential to be zero.

The electromagnetic vector potential inside the box must satisfy two conditions: (1) $\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A} = \boldsymbol{0}$. (2) For any closed path C that encircles the "hole" in the box, $\oint_C \boldsymbol{A} \cdot d\boldsymbol{l} = \int_S \boldsymbol{\nabla} \times \boldsymbol{A} \cdot d\boldsymbol{S} = \int_S \boldsymbol{B} \cdot d\boldsymbol{S} = \Phi$, the flux through the hole. These conditions are met by $A_\rho = A_z = 0$, $A_\phi = \Phi/(2\pi\rho)$, which also obeys the Coulomb gauge condition, $\boldsymbol{\nabla} \cdot \boldsymbol{A} = 0$.

Inside the box, the particle's Hamiltonian is

$$\begin{aligned} H &= -\frac{\hbar^2}{2M}\nabla^2 + \frac{i\hbar q}{2Mc}\left[2\boldsymbol{A}\cdot\boldsymbol{\nabla} + (\boldsymbol{\nabla}\cdot\boldsymbol{A})\right] + \frac{q^2}{2Mc^2}\boldsymbol{A}\cdot\boldsymbol{A} \\ &= -\frac{\hbar^2}{2M}\left(\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2} + \frac{\partial^2}{\partial z^2}\right) + \frac{i\hbar q\Phi}{2\pi Mc\rho^2}\frac{\partial}{\partial\phi} + \frac{q^2\Phi^2}{8\pi^2 Mc^2\rho^2}. \end{aligned}$$

The stationary states (which must vanish on the surface of the box) satisfy

$$H\psi_{n,m,k_z} = \varepsilon(n,m,k_z)\,\psi_{n,m,k_z}, \quad \text{where} \quad \psi_{n,m,k_z}(\rho,\phi,z) = R_n(\rho)e^{im\phi}\sin k_z z.$$

Here n, m, and lk_z/π are all integers, and R_n is a solution of

$$R_n'' + R_n'/\rho + (\alpha^2 - \nu^2/\rho^2)R_n = 0, \tag{1}$$

with

$$\alpha = \sqrt{2M\varepsilon/\hbar^2 - k_z^2}, \quad \nu = m - \Phi q/hc.$$

The transformation $\rho \to \tilde{\rho}/\alpha$ converts Eq. (1) into Bessel's equation. Thus,

$$R_n(\rho) = A_n J_\nu(\rho/\alpha_n) + B_n N_\nu(\rho/\alpha_n),$$

where α_n (and ε) is determined, along with B_n/A_n, by the conditions $R_n(a) = R_n(b) = 0$.

The key point is that the values of ε that satisfy the boundary conditions depend on the enclosed flux Φ (via ν), even though $\boldsymbol{B} = \boldsymbol{0}$ throughout the region in which the wave function is non-vanishing. Possible interpretations are that the electromagnetic vector potential is physically significant (even though its observable effects must be gauge-invariant), or that the magnetic field acts non-locally.