Reduction of a Dyadic to a Sum of Spherical Tensor Operators

- As an example of a reducible rank-2 tensor, consider the dyadic

$$T_{jk} = U_j V_k,$$

where $\boldsymbol{U}$ and $\boldsymbol{V}$ are vector operators satisfying

$$[U_j, J_k] = i\hbar \sum_l \epsilon_{jkl} U_l, \qquad [V_j, J_k] = i\hbar \sum_l \epsilon_{jkl} V_l.$$

It is an identity that

$$U_j V_k = \left[\tfrac{1}{3}\boldsymbol{U}\cdot\boldsymbol{V}\,\delta_{jk}\right] + \left[\tfrac{1}{2}(U_j V_k - U_k V_j)\right] + \left[\tfrac{1}{2}(U_j V_k + U_k V_j) - \tfrac{1}{3}\boldsymbol{U}\cdot\boldsymbol{V}\,\delta_{jk}\right].$$

The three parts in square brackets have very different properties:

$$\begin{aligned} \tfrac{1}{3}\boldsymbol{U}\cdot\boldsymbol{V}\,\delta_{jk} &= \text{a multiple of the identity matrix,} \\ \tfrac{1}{2}(U_j V_k - U_k V_j) &= \tfrac{1}{2}\sum_l \epsilon_{jkl}(\boldsymbol{U}\times\boldsymbol{V})_l, \text{ an asymmetric (traceless) tensor,} \\ \tfrac{1}{2}(U_j V_k + U_k V_j) - \tfrac{1}{3}\boldsymbol{U}\cdot\boldsymbol{V}\,\delta_{jk} &= \text{a symmetric, traceless tensor.} \end{aligned}$$

We will show below that each of these parts is associated with a spherical tensor operator, each of which transforms as a different irreducible representation of the rotation group SO(3).

- Associated with the vector operators $\boldsymbol{U}$ and $\boldsymbol{V}$ are a pair of rank-1 spherical tensor operators $U_q^{(1)}$ and $V_q^{(1)}$, where

$$U_1^{(1)} = \tfrac{1}{\sqrt{2}}(-U_x - iU_y), \quad U_0^{(1)} = U_z, \quad U_{-1}^{(1)} = \tfrac{1}{\sqrt{2}}(U_x - iU_y),$$

and similarly for $\boldsymbol{V}$.

These two rank-1 spherical tensor operators can be combined to form three compound spherical tensor operators:

1. A rank-0 operator,

$$T_0^{(0)} = \tfrac{1}{3}\left(U_1^{(1)}V_{-1}^{(1)} - U_0^{(1)}V_0^{(1)} + U_{-1}^{(1)}V_1^{(1)}\right) = -\tfrac{1}{3}\boldsymbol{U}\cdot\boldsymbol{V},$$

   which is associated with the scalar $S = \boldsymbol{U}\cdot\boldsymbol{V}$.

2. A rank-1 operator $T_q^{(1)}$, with

$$T_1^{(1)} = \tfrac{1}{\sqrt{2}}\left(U_1^{(1)}V_0^{(1)} - U_0^{(1)}V_1^{(1)}\right) = \tfrac{i}{2}(\boldsymbol{U}\times\boldsymbol{V})_x + \tfrac{1}{2}(\boldsymbol{U}\times\boldsymbol{V})_y = -\tfrac{i}{\sqrt{2}}(U\times V)_1^{(1)},$$

   and

$$T_q^{(1)} = -\tfrac{i}{\sqrt{2}}(U\times V)_q^{(1)},$$

   which is associated with the vector $\boldsymbol{W} = \boldsymbol{U}\times\boldsymbol{V}$.

3. A rank-2 operator $T_q^{(2)}$, where

$$\begin{aligned}
T_{\pm 2}^{(2)} &= U_{\pm 1}^{(1)} V_{\pm 1}^{(1)}, \\
T_{\pm 1}^{(2)} &= \tfrac{1}{\sqrt{2}} \left(U_{\pm 1}^{(1)} V_0^{(1)} + U_0^{(1)} V_{\pm 1}^{(1)} \right), \\
T_0^{(2)} &= \tfrac{1}{\sqrt{6}} \left(U_1^{(1)} V_{-1}^{(1)} + 2U_0^{(1)} V_0^{(1)} + U_{-1}^{(1)} V_1^{(1)} \right),
\end{aligned}$$

which is associated with the quadrupole tensor

$$Q_{jk} = \tfrac{1}{2}(U_j V_k + U_k V_j) - \tfrac{1}{3} \boldsymbol{U} \cdot \boldsymbol{V} \delta_{jk}.$$

- It is obvious that

$$\tfrac{1}{3} \boldsymbol{U} \cdot \boldsymbol{V} \delta_j k = \begin{pmatrix} -T_0^{(0)} & 0 & 0 \\ 0 & -T_0^{(0)} & 0 \\ 0 & 0 & -T_0^{(0)} \end{pmatrix}.$$

After some tedious algebra, one can also show that

$$\tfrac{1}{2}(U_j V_k - U_k V_j) = \frac{1}{2} \begin{pmatrix} 0 & i\sqrt{2} T_0^{(1)} & T_1^{(1)} + T_{-1}^{(1)} \\ -i\sqrt{2} T_0^{(1)} & 0 & -i \left(T_1^{(1)} - T_{-1}^{(1)} \right) \\ -\left(T_1^{(1)} + T_{-1}^{(1)} \right) & i \left(T_1^{(1)} - T_{-1}^{(1)} \right) & 0 \end{pmatrix}$$

and

$$\tfrac{1}{2}(U_j V_k + U_k V_j) - \tfrac{1}{3} \boldsymbol{U} \cdot \boldsymbol{V} \delta_{jk} =$$

$$\frac{1}{2} \begin{pmatrix} T_2^{(2)} + T_{-2}^{(2)} - \sqrt{\tfrac{2}{3}} T_0^{(2)} & -i \left(T_2^{(2)} - T_{-2}^{(2)} \right) & -T_1^{(2)} + T_{-1}^{(2)} \\ -i \left(T_2^{(2)} - T_{-2}^{(2)} \right) & -T_2^{(2)} - T_{-2}^{(2)} - \sqrt{\tfrac{2}{3}} T_0^{(2)} & i \left(T_1^{(2)} + T_{-1}^{(2)} \right) \\ -T_1^{(2)} + T_{-1}^{(2)} & i \left(T_1^{(2)} + T_{-1}^{(2)} \right) & \sqrt{\tfrac{8}{3}} T_0^{(2)} \end{pmatrix}.$$

Each of these parts is irreducible, i.e., the nonzero elements of each matrix form an invariant subspace under the action of $U[R]$.