PHY 6646 Spring 2003 – Mid-Term Exam 1

Instructions: Attempt both questions, each of which is worth 50 points. The maximum score for each part of each question is shown in square brackets. To gain full credit you should explain your reasoning and show all working. Please write neatly and remember to include your name on the front page of your answers.

Please read carefully: During this exam, you may use (and quote results from) (i) Shankar's *Principles of Quantum Mechanics* (ii) lecture notes from this course, and (iii) homework solutions from this course. You are not permitted to consult any other books, notes, or papers, or to communicate with anyone other than the proctor. In accordance with the UF Honor Code, by turning in this exam to be graded, you affirm the following pledge: *On my honor, I have neither given nor received unauthorized aid in doing this assignment.*

1. Consider the one-dimensional potential

$$V(x) = \begin{cases} Ua\,\delta(x) & \text{for } |x| \le a, \\ +\infty & \text{for } |x| > a, \end{cases}$$

where U is a positive energy and a a positive length. If $U = 0$, then this reduces to the problem of a particle in a box, which has stationary state wave functions

$$\phi_n(x) = C_n \sin\left[\frac{(n+1)\pi(x+a)}{2a}\right], \quad n = 0,\, 1,\, 2,\, \ldots$$

(a) [5 points] Which (if any) of the wave functions $\phi_n(x)$ are also stationary states of the potential $V(x)$ when $U > 0$?

(b) [20 points] Using the functions $\phi_n(x)$ as a family of trial wave functions, find the lowest possible upper bound $E_b(U)$ on the true ground-state energy $E_0(U)$.

The true ground state wave function can be written $\psi_0(x) = C_0 \sin k(a - |x|)$, where k is a solution of

$$\tan ka = -ka/u, \quad \text{with} \quad u = ma^2 U/\hbar^2. \tag{1}$$

(c) [15 points] Find an approximate solution to Eq. (1) valid for small U, i.e., for $u \ll 1$. [Hint: Write $ka = \frac{\pi}{2} + \varepsilon$ and find an approximate value for ε.] Hence, show that the variational bound E_b agrees with the exact result E_0 to first order in u. [It turns out that $(E_b - E_0)/E_0 \propto u^2$, but you do not need to show this.]

Sketch the ground-state wave function(s) for this limit.

(d) [10 points] Argue that the variational bound E_b approaches the exact result E_0 in the limit of large U, i.e., for $u = \infty$.

Sketch the ground-state wave function(s) for this limit.

2. A spin-$\frac{1}{2}$ degree of freedom is located in a constant magnetic field $\boldsymbol{B}$, which has Cartesian coordinates $(B_x, B_y, B_z) = (B\sin\theta\cos\phi, B\sin\theta\sin\phi, B\cos\theta)$ with $B > 0$. For the purposes of this question, you should ignore all orbital degrees of freedom.

   Suppose that $\bar{\mu}_z(0) = -\hbar\gamma/2$, where $\bar{\boldsymbol{\mu}}(t)$ denotes the expectation value of the magnetic moment $\boldsymbol{\mu} = \gamma\boldsymbol{S}$ at time t.

   (a) [8 points] Find a normalized spinor $\chi(0)$ describing the state of the spin at time $t = 0$ in the basis of eigenstates of S_z. Explain your reasoning.

   (b) [8 points] Find a normalized spinor $\chi(t)$ describing the state of the spin at time t in the basis of eigenstates of S_z.

   (c) [14 points] Calculate $\bar{\mu}_x(t)$ for this spin. Simplify your answer.

   (d) [14 points] Find two different choices of the field $\boldsymbol{B}$ (i.e., values of B, θ, and ϕ), such that $\bar{\mu}_x(T) = +\hbar\gamma/2$, where T is some fixed time. Your two choices should have different values of θ. For each θ, identify the weakest field (smallest B) that will achieve the stated goal.

   (e) [6 points] Calculate $\bar{\mu}_x(T/2)$ for each of your choices of field from part (d).