## PHY 6646 Spring 2003 – Homework 3

**Due by 5 p.m. on Friday, January 31. Partial credit will be available for solutions submitted by 5 p.m. on Monday, February 3.**

*Answer both questions. To gain maximum credit you should explain your reasoning and show all working. Please write neatly and include your name on the front page of your answers.*

1. Consider an isolated point-like particle of mass $m$ and charge $q$, situated in a static, uniform magnetic field $\boldsymbol{B}$, and described by the Hamiltonian

$$H = \frac{1}{2m}\left|\boldsymbol{P} - \frac{q}{c}\boldsymbol{A}\right|^2 + q\phi. \tag{1}$$

Any choice of gauge for the electromagnetic potential can be expressed as a local gauge transformation applied to the symmetric Coulomb gauge:

$$\boldsymbol{A}^{(\Lambda)}(\boldsymbol{r},t) = \frac{1}{2}\boldsymbol{B}\times\boldsymbol{r} - \boldsymbol{\nabla}\Lambda(\boldsymbol{r},t), \quad \phi^{(\Lambda)}(\boldsymbol{r},t) = \frac{1}{c}\frac{\partial\Lambda(\boldsymbol{r},t)}{\partial t},$$

where each gauge is specified by a different real-valued function $\Lambda(\boldsymbol{r},t)$. The real-space wave function corresponding to an abstract state $|\psi\rangle$ is also gauge-dependent; let us denote it $\psi^{(\Lambda)}(\boldsymbol{r},t)$.

(a) Substitute the coordinate-representations of $\boldsymbol{P}$, $\boldsymbol{A}^{(\Lambda)}$, and $\phi^{(\Lambda)}$ into the Hamiltonian (1), expand all parentheses, and then write the resulting expression in the form $H^{(\Lambda)} = H_0^{(\Lambda)} + H_{\boldsymbol{B}}^{(\Lambda)}$, where $H_0^{(\Lambda)}$ is the Hamiltonian for the case $\boldsymbol{B} = \boldsymbol{0}$.

(b) Find the particle's orbital magnetic moment operator, defined (by analogy with classical electromagnetism) as $\boldsymbol{\mu}_L^{(\Lambda)} = -\partial H^{(\Lambda)}(\boldsymbol{B}')/\partial\boldsymbol{B}'\big|_{\boldsymbol{B}'=\boldsymbol{0}}$. Make it resemble as closely as possible its form in the gauge $\Lambda = 0$, i.e., $\boldsymbol{\mu}_L^{(0)} = q/(2mc)\boldsymbol{L}$, where $\boldsymbol{L}$ is the orbital angular momentum.

(c) Show that, even though the orbital angular momentum operator defined in part (b) is gauge-dependent, the matrix element of $\boldsymbol{\mu}_L^{(\Lambda)}$ between any pair of states $\psi^{(\Lambda)}(\boldsymbol{r},t)$ and $\chi^{(\Lambda)}(\boldsymbol{r},t)$ is gauge-independent.

(d) Show that if the magnetic operator is given the alternative definition $\boldsymbol{\mu}_L^{(\Lambda)} = -\partial H^{(\Lambda)}(\boldsymbol{B}')/\partial\boldsymbol{B}'\big|_{\boldsymbol{B}'=\boldsymbol{B}}$, this operator can be expressed in the gauge-independent form $\boldsymbol{\mu}_L = (q/2c)\boldsymbol{R}\times\boldsymbol{V}$, where $\boldsymbol{V}$ is the velocity operator.

(e) $H_0$ found in part (a) is gauge-dependent, even though it describes a situation when no electromagnetic field is present. Explain the source of this gauge dependence. Show that the matrix element of $H_0^{(\Lambda)} - i\hbar\partial/\partial t$ between any pair of states $\psi^{(\Lambda)}(\boldsymbol{r},t)$ and $\chi^{(\Lambda)}(\boldsymbol{r},t)$ is gauge-independent.

2. Based on Ballentine Problem 12.2: The most general state operator for a spin-$\frac{1}{2}$ system has the form $\rho = \frac{1}{2}(I + \boldsymbol{p} \cdot \boldsymbol{\sigma})$, where $\sigma_j$ ($j = 1,\ 2,\ 3$) is a Pauli operator and $\boldsymbol{p}$ is the (real) polarization vector, whose length cannot exceed 1. Suppose that the system has a magnetic moment $\boldsymbol{\mu} = \frac{1}{2}\gamma\hbar\boldsymbol{\sigma}$ and is in a constant, uniform magnetic field $\boldsymbol{B}$.

   (a) Calculate the time dependence of $\rho(t)$ in the Schrödinger picture. Describe the result geometrically in terms of the variation of the vector $\boldsymbol{p}$, assuming a general value for $\boldsymbol{p}(t = 0)$.

   (b) Now suppose that the system is in thermal equilibrium at temperature $T$, so that $\rho_H \propto \exp(-H/k_B T)$. Find $\boldsymbol{p}(t)$ for this situation, and hence deduce $\langle \boldsymbol{\mu}(t) \rangle$.