PHY 6646 Spring 2003 – Homework 5

Due by 5 p.m. on Friday, February 14. Partial credit will be available for solutions submitted by 5 p.m. on Monday, February 17.

Answer all questions. To gain maximum credit you should explain your reasoning and show all working. Please write neatly and include your name on the front page of your answers.

1. Show that the WKB approximation yields the true bound-state energies of the harmonic oscillator potential, $V(x) = \frac{1}{2}m\omega^2 x^2$.

   You may find it useful to know that

   $$\int_{-1}^{1} \sqrt{1-x^2}\, dx = \int_{-\pi/2}^{\pi/2} \cos^2\theta \, d\theta = \pi/2.$$

2. Consider a particle of mass m moving in the one-dimensional potential $V(x) = -\mu x^2 + \lambda x^4$, where μ and λ are positive, real quantities.

   (a) Show that this potential describes two potential wells separated by a finite barrier, with the bottom of each well being approximately harmonic (i.e., quadratic in the displacement from the bottom of the well). Find (i) the separation d between the bottom of the two wells; (ii) the height of the energy barrier, measured from the bottom of each well, and (iii) the effective frequency ω describing small oscillations within one of the wells.

   (b) The goal of this question is to find energy eigenstates having an energy E sufficiently low that there is only a very small (but nonetheless nonzero) probability of finding the particle at any point where the potential is not harmonic. By considering the characteristic width of the wave function describing a oscillation within a single well, find an inequality satisfied by the energy eigenvalue E that places the system in this limit.

   (c) For the limit described in part (b), write down the WKB wave function in each region for which such a wave function is valid. You may express your answer in terms of the wavelength $k(x)$ and the inverse decay length $\kappa(x)$. You should use the connection formulae to ensure that your answers contain only one unknown amplitude: that characterizing the wave function in the region $x \to -\infty$. (This amplitude can, of course, be determined up to a complex phase factor by normalizing the wave function, but you need not bother with this step.)

   (d) Show that the bound states in this limit are determined by the condition $\tan\theta = \pm 2e^{\alpha}$, with $\theta = \int k(x)dx$ being the integral of the wavevector across the classically accessible region within one of the potential wells, and $\alpha = \int \kappa(x)dx$ being the integral of the inverse decay length across the classically forbidden region between the two wells.

   (e) By evaluating θ explicitly, assuming that the classically accessible region of each potential well is described by a pure-quadratic potential, show that the low-lying eigenvalues come in pairs with an approximate energy splitting $\hbar\omega e^{-\alpha}/\pi$. What is the principal difference between the two wave functions corresponding to each pair of eigenvalues?

3. Use the WKB approximation to find the three lowest energy eigenvalues for a particle of mass m moving in the one-dimensional potential

$$V(z) = \begin{cases} \infty & \text{for } z < 0, \\ mgz & \text{for } z \geq 0. \end{cases}$$

Compare your answers with the exact eigenvalues found in Homework 2, Question 2: $(\hbar^2 m g^2)^{-1/3} E = 1.856,\ 3.245,\ 4.382$.

4. This problem deals with the WKB treatment of particles traveling over (rather than tunneling through) a potential barrier. Suppose that a particle of mass m is moving in a one-dimensional potential $V(x)$ that (i) is continuous and has a continuous first derivative everywhere, and (ii) satisfies $V(x) = V_L$ for $x \leq x_L$ and $V(x) = V_R$ for $x \geq x_R$.

Consider a stationary state of energy E sufficiently high that for all x, $E > V(x)$, $|k'(x)| \ll |k(x)|^2$, and $|k''(x)| \ll |k(x)|^3$. Let the stationary state wave function be written

$$\psi(x \leq x_L) = \frac{A_L}{\sqrt{k_L}} e^{ik_L x} + \frac{B_L}{\sqrt{k_L}} e^{-ik_L x},$$

and

$$\psi(x \geq x_R) = \frac{A_R}{\sqrt{k_R}} e^{ik_R x} + \frac{B_R}{\sqrt{k_R}} e^{-ik_R x}.$$

(a) Use the WKB approximation to find A_R and B_R in terms of A_L and B_L.

(b) Find the transmission and reflection coefficients, $T(E)$ and $R(E)$, for a flux of particles incoming from $x = -\infty$. Compare your results qualitatively with what you would expect to be the true behaviors. In what range of E is there a significant discrepancy between the WKB and true results? In what range of E do these sets of results converge?