PHY 6646 Spring 2004 – Homework 1

Due by 5 p.m. on Friday, January 23. No credit will be available for homework submitted after 5 p.m. on Monday, January 26.

Answer all questions. Please write neatly and include your name on the front page of your answers. You must also clearly identify all your collaborators on this assignment. To gain maximum credit you should explain your reasoning and show all working.

1. Derive the "global" definition of a vector operator,

$$U^{-1}[R]\, V_j\, U[R] = \sum_k R_{jk} V_k,$$

from the "local" definition:

$$[V_j, J_k] = i\hbar \sum_l \epsilon_{jkl} V_l.$$

You may find it useful to employ (i) the Baker-Hausdorff lemma,

$$e^{\lambda\Omega}\Lambda e^{-\lambda\Omega} = \Lambda + \lambda[\Omega,\Lambda] + \frac{\lambda^2}{2!}[\Omega,[\Omega,\Lambda]] + \frac{\lambda^3}{3!}[\Omega,[\Omega,[\Omega,\Lambda]]] + \ldots,$$

where Ω and Λ are operators, and λ is a constant, and (ii) the fact that the rotation matrix for a rotation $\boldsymbol{\omega}$ can be expressed

$$R(\boldsymbol{\omega}) = \exp\left(-i\boldsymbol{\omega}\cdot\bar{\boldsymbol{J}}\right),$$

where $\bar{J}_j$ is a 3×3 matrix having elements

$$[\bar{J}_j]_{kl} = -i\epsilon_{jkl}.$$

2. Consider a set of angular momentum eigenstates $|\alpha, j, m\rangle$, where α labels all quantum numbers other than j and m, and let $\boldsymbol{V}$ be a vector operator. Using the Wigner-Eckart theorem, plus the recursion relations for Clebsch-Gordan coefficients (property 3 on the third page of the PHY 6645 handout "Addition of Two Arbitrary Angular Momenta" (http://www.phys.ufl.edu/~kevin/teaching/6645/03fall/addition.pdf), find

(a) $\dfrac{\langle\alpha', j, j-1|V_x|\alpha, j, j\rangle}{\langle\alpha', j, j|V_z|\alpha, j, j\rangle}$; (b) $\dfrac{\langle\alpha', j, j-1|V_z|\alpha, j, j-1\rangle}{\langle\alpha', j, j|V_z|\alpha, j, j\rangle}$.

Your answers should not contain any unevaluated Clebsch-Gordan coefficients.

3. This problem addresses a simple model for an electron in a MOSFET (metal-oxide-semiconductor field effect transistor). The electron is considered to be confined by insulating barriers to the interior of a cylinder of radius R. A uniform external electric field E is applied parallel to the rotational symmetry axis of the cylinder (assumed to be the z axis), causing the electron to have a potential energy

$$V(x,y,z) = \begin{cases} eEz & \text{for } x^2+y^2 < R^2 \text{ and } z > 0, \\ \infty & \text{otherwise,} \end{cases}$$

where $-e$ is the charge of the electron. (Note that the finite height of the cylinder is assumed not to play a significant role in the problem: the particle is confined at large z by the electric field, not the insulating barrier, so we can treat the confining box as being semi-infinite along the z direction.)

Due to the crystal structure of the semiconductor, an electron has different effective masses for motion parallel to and perpendicular to the z axis. Thus, the Schrödinger equation for this problem is

$$\left[\frac{\hbar^2}{2m_\perp}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)+\frac{\hbar^2}{2m_\parallel}\frac{\partial^2}{\partial z^2}+\varepsilon-V(x,y,z)\right]\psi(x,y,z)=0,$$

where ε is the energy eigenvalue.

(a) Show that the Schrödinger equation is separable if one transforms to cylindrical polar coordinates, i.e., the stationary states can be written in the form $\psi(\rho,\phi,z)=R(\rho)\Phi(\phi)Z(z)$. Show, furthermore, that the energy eigenvalues can be written $\varepsilon=\varepsilon_{R\Phi}+\varepsilon_Z$, where $E_{R\Phi}$ comes from solving the Schrödinger equation for $R(\rho)\Phi(\phi)$, and E_Z comes from solving the Schrödinger equation for $Z(z)$.

(b) Show that $\varepsilon_{R\Phi}$ takes discrete values, and find the lowest four distinct values of $\varepsilon_{R\Phi}$.

(c) Show that ε_z takes discrete values, and find the lowest four distinct values of ε_z.

(d) List the four lowest distinct values of ε, and give the degeneracy of the each value (assuming that there is no accidental degeneracy arising from the particular choice of R, E, and the effective masses).

Note: There are two independent energy scales in this problem—one arising from the finite radius of the device, and the other connected with the applied electric field. For completeness, please list enough states to ensure that you include the four lowest distinct values of ε for any strength of the electric field.