PHY 6646 Spring 2004 – Homework 2

Due by 5 p.m. on Friday, January 30. No credit will be available for homework submitted after 5 p.m. on Monday, February 2.

Answer both questions. Please write neatly and include your name on the front page of your answers. You must also clearly identify all your collaborators on this assignment. To gain maximum credit you should explain your reasoning and show all working.

1. *Two-dimensional electrons in a perpendicular magnetic field:* Consider an electron of mass m and charge $-e$ restricted to move in the x–y plane.[1] Suppose that the electron is confined to a region described by coordinates $0 < x < L_x$ and $0 < y < L_y$, and that it experiences a uniform magnetic field $\boldsymbol{B} = B\hat{\boldsymbol{z}}$ ($B > 0$).

   (a) Working in the Landau gauge $\phi = 0$, $\boldsymbol{A} = Bx\hat{\boldsymbol{y}}$, and assuming periodic boundary conditions, find the stationary-state wave functions $\psi(x, y)$ and their energies ε. [Throughout this question, you may ignore the ground-state energy of $V(z)$.]

   (b) Find the density of states $g(\varepsilon)$.

   Suppose now that a gas of N electrons occupies the region $0 < x < L_x$, $0 < y < L_y$. We will ignore both the spin of each electron and the Coulomb interaction between different electrons.[2] Due to the Pauli exclusion principle, the ground state of the N-electron system is found by putting one electron in each of the N lowest-energy single-particle stationary states.

   (c) Show that the ground-state energy of the electron gas is

   $$\mathcal{E}_0(N, B) = \frac{\pi\hbar^2}{mL_xL_y}\left[(2n_B + 1)N - n_B(n_B + 1)N_B\right]N_B,$$

   where N_B is the number of electrons that can fit into one Landau level and $n_B = \lfloor N/N_B \rfloor$ is the number of fully-filled Landau levels. Here, $\lfloor x \rfloor$ is the greatest integer less than or equal to x.

   (d) Sketch or plot $\mathcal{E}_0(N, B)/\mathcal{E}_0(N_B, B)$ vs N/N_B at fixed B over the range $0 \leq N/N_B \lesssim 5$.

   (e) Sketch or plot $[\mathcal{E}_0(N, B)/\mathcal{E}_0(N, 0)] - 1$ vs N/N_B at fixed N over the range $0 \leq N/N_B \lesssim 10$. You should find a striking, non-monotonic behavior.[3]

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[1] This might be accomplished by imposing a strong attractive potential $V(z)$, so that all the states of interest correspond to the ground state of $V(z)$. Such a situation can arise inside a semiconductor when an electric field is used to trap charge carriers near an insulating barrier (see Homework 1). Then the wave function can be factorized $\psi(x, y, z) = \psi(x, y)\psi_0(z)$, where $\psi_0(z)$ is the ground-state wave function of $V(z)$ and $\psi(x, y)$ is the solution to the effective two-dimensional problem considered in this question.

[2] The second assumption is largely justified if there are positive charges nearby that screen the Coulomb interaction. This situation can be realized in semiconductor devices such as MOSFETs.

[3] This non-monotonicity is the root of the "quantum oscillations" observed in the B dependence of the magnetization and the resistance of pure metals at low temperatures: the *de Haas-van Alphen* and *Shubnikov-de Haas* effects, respectively.

This simple model can be used to provide a crude explanation of the *integer quantum Hall effect*. Consider the in-plane electrical conductivity of the electron gas in the presence of an electrical field $\boldsymbol{E} = E\hat{\boldsymbol{x}}$ as well as the magnetic field $\boldsymbol{B} = B\hat{\boldsymbol{z}}$. Classically, one expects the field to induce current densities

$$j_x = \sigma_{xx}E, \qquad j_y = \sigma_{yx}E,$$

where the longitudinal conductivity is

$$\sigma_{xx} = \sigma_{yy} = \frac{ne^2\tau/m}{1 + (\omega_c\tau)^2},$$

and the transverse (Hall) conductivity is

$$\sigma_{yx} = \sigma_{xy} = \omega_c\tau\,\sigma_{xx}.$$

Here, n is the electron number density, τ is the mean time between consecutive scattering events for a single electron, and $\omega_c = eB/mc$ is the cyclotron frequency.

Quantum mechanics enters the picture above mainly through its effect on the scattering time τ. Each scattering event knocks an electron from one single-particle stationary state (the "initial" state) of energy ε_i to another (the "final" state) of energy ε_f. Since the Pauli exclusion principle forbids double occupancy of any single-particle state, the final state must be unoccupied prior to the scattering event. Furthermore, to ensure that the event is energetically possible at temperature T, it is necessary that $\varepsilon_f - \varepsilon_i \lesssim O(k_BT)$. For $k_BT \ll \hbar\omega_c$ (the splitting between adjacent Landau levels), there are no empty single-particle states in Landau levels below the highest occupied level. (The highest occupied level is the Landau level that contains the electron of highest single-particle energy in the ground state of the electrons gas.) At the same time, states in Landau levels above the highest occupied level are too far up in energy to be thermally accessible. This means that scattering can take place only if there are empty states in the highest occupied Landau level.

(f) Based on the preceding discussion, show that for fixed N, there are values of the magnetic field B_ν, $\nu = 1$, 2, 3, ... at which τ must be infinite. Show that at $B = B_\nu$, $\sigma_{xx} = 0$ and $\sigma_{xy} = \nu e^2/h$, i.e., the Hall conductance is quantized.

(g) Calculate the components of the 2×2 resistivity tensor, $\rho = \sigma^{-1}$, for $B = B_\nu$. You should find that ρ_{xy} takes quantized values.[4]

2. Shankar Exercise 14.4.3.

Before tackling this question, you may find it helpful to read the section "Paramagnetic Resonance" on pp. 392–394 of Shankar.

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[4]In real systems, disorder causes the quantization of the Hall resistance (and conductance) to extend over a finite range of magnetic fields around $B = B_\nu$. These "Hall plateaus" in ρ_{xy} provide the most accurate known experimental determination of e^2/h (with an error of order 1 part in 10^8), and are now employed in the fundamental standard for electrical resistance.