PHY 6646 Spring 2004 – Homework 4

Due by 5 p.m. on Friday, February 13. No credit will be available for homework submitted after 5 p.m. on Monday, February 16.

Answer all questions. Please write neatly and include your name on the front page of your answers. You must also clearly identify all your collaborators on this assignment. To gain maximum credit you should explain your reasoning and show all working.

1. Consider the one-dimensional cubic potential

$$V(x) = V_0 |x/a|^3,$$

where V_0 and a are positive, real constants.

   (a) Use the WKB method to approximate the allowed bound-state energies for this potential.

   (b) Compare the WKB value for the ground state energy with a variational estimate obtained using a suitable trial wave function of your own choosing.

   (c) Show that the connection formulae are valid only for stationary states whose energies exceed a threshold value $E_{\min}$. Obtain an expression for $E_{\min}$, and determine whether or not the connection formulae are valid for the WKB ground-state.

   Note: The definition if $E_{\min}$ is not unique. Any reasonable definition will receive full credit.

   (d) Estimate the number of bound states of the truncated cubic potential

$$V(x) = \begin{cases} V_0 |x/a|^3 & \text{for } |x| \leq a, \\ V_0 & \text{for } |x| \geq a. \end{cases}$$

   (e) Show that the WKB method predicts that there is no bound state unless V_0 exceeds a nonzero minimum value $V_{\min}$ and find this value.

   Note: The existence of $V_{\min}$ is an artifact of the approximations inherent in the method, because there is a general theorem (see Shankar Exercise 5.5.5) that every attractive potential in one dimension has at least one bound state.

2. Consider the one-dimensional potential $V(x) = U\left(e^{-2x/d} - 2e^{-x/d}\right)$, where U and d are positive, real quantities.

   (a) Write down the WKB quantization condition governing the bound states of this potential. (Label these states $n = 0, 1, 2, \ldots$ in order of increasing energy E_n.)

   (b) Transform the integration variable entering the WKB quantization condition from x to $y = e^{-x/d}$, and evaluate the resulting integral to obtain a closed-form expression for E_n.

   Hint: You should be able to make use of the fact that

$$\int_a^b \frac{dx}{x} \sqrt{(b-x)(x-a)} = \frac{\pi}{2} \left(\sqrt{a} - \sqrt{b}\right)^2.$$

(c) Estimate the number of bound states supported by the potential $V(x)$.

(d) Compare the WKB energies with those obtained by approximating $V(x)$ as a harmonic potential: $V(x) \approx V_0 + \frac{1}{2}m\omega^2(x-x_0)^2$, where V_0, ω, and x_0 are deduced by fitting the bottom of the potential well.

3. Consider the application of the WKB method to the tunneling of particles through a potential barrier described by a smoothly varying one-dimensional potential $V(x)$. Suppose that the WKB wave functions in the classically accessible regions to the left ($j = L$) and right ($j = R$) of the barrier are written in the form

$$\psi_j(x) = \frac{A_j}{\sqrt{k(x)}} \exp\left[i\int_{a_j}^{x} k(x')dx'\right] + \frac{B_j}{\sqrt{k(x)}} \exp\left[-i\int_{a_j}^{x} k(x')dx'\right],$$

where a_L [a_R] is the coordinate of the classical turning point on the left [right] side of the barrier.

Prove the formula given in class:

$$\begin{pmatrix} A_R \\ B_R \end{pmatrix} = \begin{pmatrix} \cosh\alpha & -i\sinh\alpha \\ i\sinh\alpha & \cosh\alpha \end{pmatrix} \begin{pmatrix} A_L \\ B_L \end{pmatrix},$$

where

$$\alpha = \int_{a_L}^{a_R} \kappa(x)dx + \ln 2.$$