PHY 6646 Spring 2004 – Homework 5

Due by 5 p.m. on Monday, March 1. No credit will be available for homework submitted after 5 p.m. on Friday, March 5.

Answer all questions. Please write neatly and include your name on the front page of your answers. You must also clearly identify all your collaborators on this assignment. To gain maximum credit you should explain your reasoning and show all working.

1. A harmonic oscillator is described by the Hamiltonian $H_0 = P^2/(2m) + m\omega^2 X^2/2$. Calculate the leading non-vanishing correction to the energy eigenvalues produced by a cubic perturbation $H_1 = \alpha x^3$.

2. A particle of mass m is confined to a one-dimensional box by the potential

$$V(x) = \begin{cases} V_1 \cos(\pi x/b) & \text{for } |x| < a, \\ \infty & \text{otherwise.} \end{cases}$$

   (a) Calculate the energy eigenvalues E_n ($n = 1, 2, 3, \ldots$) to first order in the small quantity $32ma^2|V_1|/h^2$.

   (b) A crude description of the potential experienced by an electron inside a crystalline solid can be obtained by considering the special case $a = Nb$, where N is a (positive) integer. Show that in this case, exactly one energy E_n contains a term that is first-order in V_1.

3. Modified from Ballentine Problem 10.14: The three-fold degenerate energy level of the hydrogen atom, with eigenvectors $|n = 2, l = 1, m = \pm 1, 0\rangle$, is subjected to a perturbation of the form $H_1 = bxz$, where b is a real scalar. Use degenerate perturbation theory to determine the zero-order eigenvectors and the splitting of the energy levels to first order in b.

4. The fine structure of hydrogen: The energy levels of the hydrogen atom deduced from the Coulomb potential [the E_n's given by Shankar (13.1.16)] must be corrected for a number of physical effects, which produce a hierarchy of measurable energy shifts: the fine structure ($\Delta E_n/E_n$ of order α^2), the Lamb shift (of order α^3), and the hyperfine splitting (of order α^4), where $\alpha = e^2/\hbar c \approx 1/137$ is the fine-structure constant. Here we focus on the largest of these shifts.

   The fine structure arises from relativistic effects, and is most naturally obtained using the Dirac equation (see Shankar Section 20.2), which predicts the energy levels of the hydrogen atom to be given by Eq. (20.2.40). However, it is also possible to calculate the fine structure within nonrelativistic quantum mechanics, treating the relativistic effects as (given) perturbations of the pure Coulomb potential (see Shankar Section 17.3).

   (a) There is no work to be turned in for this part. Read Shankar's discussion of fine structure on pp. 466–470. Note in particular that the total fine-structure perturbation can be written

$$H_1 = -\frac{1}{2mc^2}\left(H_0 + \frac{e^2}{r}\right)^2 + \frac{e^2 \mathbf{L}\cdot\mathbf{S}}{2m^2c^2r^3},$$

which commutes with the total angular momentum operators J^2 and J_z, as well as with L^2 and S^2. Therefore, it makes sense to work in a basis of states labeled by five quantum numbers: n, j, m_j, l, and s. Through first order in the fine-structure perturbation, the energies [given in Shankar Eq. (17.3.22)] turn out to depend only on n and j.

(b) Do Shankar Exercise 17.3.4.

5. The weak-field Zeeman effect in hydrogen: Consider the application of a weak magnetic field $\mathbf{B} = B\hat{\mathbf{z}}$ to a hydrogen atom, so that there is an additional Hamiltonian term (the Zeeman term):

$$H_Z = -\gamma(\mathbf{L} + g\mathbf{S}) \cdot \mathbf{B} = -\gamma B[J_z + (g-1)S_z],$$

where $\gamma = -e/2mc$ (e being assumed to be positive).

(a) Within what range of B would you expect it to be valid to treat H_Z as a perturbation of the fine structure considered in the previous question?

(b) In order to perturb about the fine structure energies, it is necessary to calculate matrix elements of S_z between states $|n, j, m_j, l, s\rangle$. [Of course, $s = \frac{1}{2}$ always in the hydrogen atom. Nonetheless, keep s as an arbitrary (integer or half-integer) quantity in this part of the question.] The Wigner-Eckart theorem (remember that?) comes in useful here. $\mathbf{S}$ is a vector operator, and hence can be used to construct a rank-1 spherical tensor. Apply Shankar (15.3.19) and the fact that $\mathbf{J}\cdot\mathbf{S} = \frac{1}{2}[\mathbf{J}^2 + \mathbf{S}^2 - (\mathbf{J}-\mathbf{S})^2]$ to show that the first-order energy shift can be written

$$\langle n, j, m_j, l, s|H_Z|n, j, m_j, l, s\rangle = g_{\text{eff}}\mu_B B m_j,$$

where $\mu_B = e\hbar/2mc$ is the Bohr magneton. Provide an explicit expression for the "Landé g-factor" g_{eff} as a function of g, j, l, and s.

(c) Calculate an approximation to the total ground state energy of hydrogen as a function of the field strength B in the range you identified in part (a).