PHY 6646 Spring 2004 – Homework 9

Due by 5 p.m. on Wednesday, April 21. No credit will be available for homework submitted after 5 p.m. on Friday, April 23.

Answer all questions. Please write neatly and include your name on the front page of your answers. You must also clearly identify all your collaborators on this assignment. To gain maximum credit you should explain your reasoning and show all working.

1. Two particles, each of mass m, are confined to a two-dimensional square box of sides L. For each of the three cases (i)–(iii) below, determine how many distinct states of the system have an energy $E \leq 5\pi^2\hbar^2/mL^2$. You should indicate how you arrived at your total number of states. It is not necessary to write down the wave function for each state.

   (a) The particles are distinguishable and spinless.

   (b) The particles are indistinguishable, spinless bosons.

   (c) The particles are indistinguishable, spin-$\frac{1}{2}$ fermions.

2. Suppose that a system of identical fermions has available just three single-particle states, denoted $|\alpha\rangle$, $\alpha = 1, 2, 3$. Simplify each of the expressions below so that, wherever possible, it is proportional to a many-particle basis state $|n_1, n_2, n_3\rangle$:

   (a) $c_3|1,0,1\rangle$;

   (b) $c_1^\dagger|1,0,1\rangle$;

   (c) $c_2 c_1^\dagger c_2^\dagger|0\rangle$.

3. Standardize the following products of bosonic (a) or fermionic (c) creation and annihilation operators. "Standardize" means (i) reduce the number of operators in each product to the minimum possible, e.g., by eliminating terms such as $c_\alpha c_\alpha$; (ii) place all creation operators to the **left** of all annihilation operators; (iii) among the creation operators, place those for low-index single particle states to the **left** of those for high-index states; and (iv) among the annihilation operators, place those for low-index single particle states to the **right** of those for high-index states.

   (a) $a_1 a_2^\dagger a_1 a_3$;

   (b) $c_1 c_2^\dagger c_1 c_3$;

   (c) $c_1^\dagger c_2 c_1 c_1^\dagger c_3 c_2^\dagger c_1$;

   (d) $a_2 a_2^\dagger a_2^\dagger a_2 a_2^\dagger$.

4. Show that the field operators obey

$$[\psi_\sigma(\mathbf{r}), \psi_{\sigma'}(\mathbf{r}')]_\pm = 0, \quad [\psi_\sigma^\dagger(\mathbf{r}), \psi_{\sigma'}^\dagger(\mathbf{r}')]_\pm = 0, \quad [\psi_\sigma(\mathbf{r}), \psi_{\sigma'}^\dagger(\mathbf{r}')]_\pm = \delta(\mathbf{r}-\mathbf{r}')\delta_{\sigma,\sigma'},$$

   where "+" applies to fermions ($[A,B]_+$ being the anticommutator of A and B) and "−" applies to bosons ($[A,B]_-$ being the commutator of A and B).

5. Consider the spontaneous decay of the hydrogen atom, as discussed pp. 517–520 of Shankar. Suppose that it is known that the atom begins in the state $|2, 1, 1\rangle$.

   (a) Calculate the differential rate of spontaneous decay of the hydrogen atom with the emission of a photon into solid angle $d\Omega$.

   (b) Integrate the differential decay rate over all angles to obtain the total rate of spontaneous decay. Verify that your answer agrees with that [Shankar (18.5.89)] obtained by averaging over initial states $|2, 1, m\rangle$.