PHZ 7427 Spring 2011 – Homework 1

Due by the start of class on Monday, February 7. After that, the assignment may be submitted for 75% credit until the start of class on Monday, February 14.

Answer all four questions. Please write neatly and include your name on the front page of your answers. You must also clearly identify all your collaborators on this assignment. To gain maximum credit you should explain your reasoning and show all working.

1. Standardize the following products of bosonic (a) or fermionic (c) creation and annihilation operators. For the purpose of this question, "standardize" means (i) *normal-order* the operators by permuting all creation operators to lie to the left of all annihilation operators; (ii) permute any creation operators to lie in order of ascending state index α; (iii) permute any annihilation operators to lie in order of descending α; (iv) rewrite any product of identical bosonic operators in power form such as $(a_\alpha)^2$ or $(a_\alpha^\dagger)^5$, and eliminate any product of identical fermionic operators using $c_\alpha c_\alpha = c_\alpha^\dagger c_\alpha^\dagger = 0$.

   (a) $a_1 a_2^\dagger a_1 a_3$; (b) $a_2 a_1^\dagger a_2^\dagger a_1 a_2^\dagger$; (c) $c_1^\dagger c_3^\dagger c_2^\dagger c_1^\dagger$; (d) $c_1^\dagger c_2 c_1 c_1^\dagger c_3 c_2^\dagger c_1$.

2. Complete the proof outlined on page 6 of the Ch. 1 notes that

   $$[\psi_\sigma(\mathbf{r}), \psi_{\sigma'}(\mathbf{r}')]_\pm = 0, \quad [\psi_\sigma^\dagger(\mathbf{r}), \psi_{\sigma'}^\dagger(\mathbf{r}')]_\pm = 0, \quad [\psi_\sigma(\mathbf{r}), \psi_{\sigma'}^\dagger(\mathbf{r}')]_\pm = \delta(\mathbf{r}-\mathbf{r}')\delta_{\sigma,\sigma'},$$

   where "+" applies to fermions ($[A,B]_+$ being the anticommutator of A and B) and "−" applies to bosons ($[A,B]_-$ being the commutator of A and B).

3. [Based on Fetter and Walecka Problem 4.1 and Phillips Ch. 5 Problem 3.] Consider a uniform gas of electrons that interact via a screened Coulomb potential

   $$u(r) = \frac{V_0}{r}\, e^{-r/a} \quad \Leftrightarrow \quad u(q) = \frac{4\pi V_0}{q^2 + a^{-2}}.$$

   (a) Show that the single-particle wave functions $\phi_{\mathbf{k}}(\mathbf{r}) = V^{-1/2} e^{i\mathbf{k}\cdot\mathbf{r}}$ solve the Hartree-Fock equations in a system of volume $V \to \infty$, and evaluate the direct and exchange terms in the energy shift $\Delta\varepsilon_{\mathbf{k}} \equiv \Delta\varepsilon_k$. Hence, find the Fermi energy ε_F.

   (b) Show that the exchange contribution to ε_F is negligible compared to the direct contribution in the weak-screening limit $k_F a \gg 1$, but that the two contributions are comparable in the strong-screening regime $k_F a \ll 1$.

   (c) Two different effective masses m_1^* and m_2^* can be defined via the expansion

   $$\begin{aligned} \varepsilon_{k+\Delta k} &= \varepsilon_k + \Delta k\, \frac{\partial \varepsilon_k}{\partial k} + \frac{1}{2}(\Delta k)^2\, \frac{\partial^2 \varepsilon_k}{\partial k^2} + \ldots \\ &= \varepsilon_k + \frac{\hbar^2 k\, \Delta k}{m_1^*} + \frac{\hbar^2 (\Delta k)^2}{2 m_2^*} + \ldots \end{aligned}$$

   where

   $$\frac{1}{m_1^*} = \frac{1}{\hbar^2 k}\frac{\partial \varepsilon_k}{\partial k} \equiv \frac{v_k}{\hbar k} \quad \left(\frac{\text{velocity}}{\text{momentum}}\right) \qquad \text{and} \qquad \frac{1}{m_2^*} = \frac{1}{\hbar^2}\frac{\partial^2 \varepsilon_k}{\partial k^2}.$$

   These masses are defined such that for free electrons, $m_1^* = m_2^* = m$.
   Express $\partial \Delta\varepsilon_k/\partial k$ and $\partial^2 \Delta\varepsilon_k/\partial k^2$ for the present system as functions of $x = k/k_F$. Hence, provide expressions for m/m_1^* and m/m_2^* for k close to k_F in each of the limits (i) $k_F a \ll 1$, and (ii) $k_F a \gg 1$.

4. A quantum-mechanical system is represented by a two-dimensional vector space spanned by an orthonormal basis $|1\rangle$ and $|2\rangle$. This system's Hamiltonian is $H = \epsilon(|2\rangle\langle 2| - |1\rangle\langle 1|)$, where $\epsilon > 0$. We also consider another Hermitian operator Ω, which can be written $\Omega = \omega_0(|1\rangle\langle 2| + |2\rangle\langle 1|)$ with $\omega_0 > 0$. Let the eigenvalues of Ω be ω_1 and ω_2 ($\omega_1 < \omega_2$) corresponding to normalized eigenkets $|\omega_1\rangle$ and $|\omega_2\rangle$.

   Consider three initial states of this system, each described by its state operator $\rho(t = 0)$:

   (a) $\rho(0) = 0.36|1\rangle\langle 1| + 0.64|2\rangle\langle 2|$.

   (b) $\rho(0) = 0.36|1\rangle\langle 1| + 0.64|2\rangle\langle 2| - 0.48|1\rangle\langle 2| - 0.48|2\rangle\langle 1|$.

   (c) $\rho(0) = 0.36|1\rangle\langle 1| + 0.64|\omega_2\rangle\langle \omega_2|$.

   For each initial state, perform the following:

   i. Determine whether the initial state is pure or mixed. If it is pure, find the corresponding normalized state vector $|\psi\rangle$.

   ii. Find the eigenvalues p_j and eigenkets $|p_j\rangle$ of the initial state operator.

   iii. Calculate the state operator $\rho(t)$ for an arbitrary time $t \geq 0$.

   iv. Use $\rho(t)$ to calculate the expectation values of H and Ω at time $t \geq 0$.