PHZ 7427 Spring 2011 – Homework 2

Due by the start of class on Wednesday, February 23. After that, the assignment may be submitted for 75% credit until the start of class on Wednesday, March 2.

Answer both questions. Please write neatly and include your name on the front page of your answers. You must also clearly identify all your collaborators on this assignment. To gain maximum credit you should explain your reasoning and show all working.

1. **The energy-loss function.** The standard theory of dielectric screening states that an external potential $\phi_{\text{ext}}(q,\omega)$ produces a total potential $\phi(q,\omega) = \phi_{\text{ext}}(q,\omega)/\epsilon(q,\omega)$ resulting from superposition of $\phi_{\text{ext}}(q,\omega)$ and an induced potential $\phi_{\text{ind}}(q,\omega) = \phi(q,\omega) - \phi_{\text{ext}}(q,\omega) = [1/\epsilon(q,\omega) - 1]\phi_{\text{ext}}(q,\omega)$. The quantity $-\text{Im}\,[1/\epsilon(q,\omega) - 1] \equiv \text{Im}\,[-1/\epsilon(q,\omega)]$ is known as the *energy-loss function* because it determines the dissipation of energy when a perturbation $\phi_{\text{ext}}(q,\omega)$ is applied to the system.

   (a) The classical theory for an electron gas with relaxation time τ and mean-free-path ℓ (see, e.g., Ashcroft and Mermin Ch. 1) gives in the limit $q\ell \ll 1$, a dielectric constant $\epsilon(q,\omega) = 1 - \omega_p^2/[\omega(\omega + i/\tau)]$. Show that within this theory, the real part of $[1/\epsilon(q,\omega) - 1]$ vanishes at $\omega = \omega_p$, while $\text{Im}\,[-1/\epsilon(q,\omega)]$ versus ω has a sharp peak centered near the same frequency. What is the approximate width of this peak in the physical limit $\omega_p\tau \gg 1$? This peak represents the strong absorption of electromagnetic energy through the creation of plasmons.

   Linear response theory shows that $1/\epsilon(q,\omega) - 1 = (4\pi e^2/q^2)\,\chi_{nn}(q,\omega)$, where $\chi_{nn}(q,\omega)$ is the double Fourier transform of the retarded density-density response function.

   (b) By introducing a complete, orthonormal basis $\{|n\rangle\}$ of exact stationary states satisfying $H_{\text{eq}}|n\rangle = E_n|n\rangle$ (as done on p. 15 of the Ch. 2 notes), provide exact expressions for the real and imaginary parts of $1/\epsilon(q,\omega)$ in the canonical ensemble at temperature $T = (k_B\beta)^{-1}$.

   (c) Using the same basis, provide an exact expression for the dynamical structure factor $S(q,\omega)$ as defined on p. 23 of the Ch. 2 notes.

   (d) By comparing your results from (b) and (c), verify the validity of the fluctuation dissipation theorem in the form $\text{Im}\,[-1/\epsilon(q,\omega)] = (4\pi^2 ne^2/\hbar q^2)(1 - e^{-\beta\hbar\omega})S(q,\omega)$.

   (e) Briefly suggest how your exact expression for $\text{Im}\,[-1/\epsilon(q,\omega)]$ could give rise to a peak of the type that you found in the classical approximation. Do not rely merely on the presence of Dirac delta functions in your exact expression for $\text{Im}\,[-1/\epsilon(q,\omega)]$ because in the limit $V \to \infty$, the sum over discrete states can generally be replaced by an integral over all states having energy ε, in which case $\text{Im}\,[-1/\epsilon(q,\omega)]$ becomes a smooth function of ω.

   (f) Show, by combining the appropriate symmetry properties of retarded linear response functions with the Kramers-Kronig relations (as defined on p. 16 of the Ch. 2 notes), that

$$\text{Re}\left[\frac{1}{\epsilon(q,\omega)}\right] = 1 + \frac{2}{\pi}\,P\!\int_0^\infty d\omega'\,\frac{\omega'}{\omega'^2 - \omega^2}\,\text{Im}\left[\frac{1}{\epsilon(q,\omega')}\right]. \tag{1}$$

Assuming that the classical theory gives the correct value of $\mathrm{Re}\,[1/\epsilon(q,\omega)]$ at very large frequencies for any q, use Eq. (1) to derive the *f-sum rule*

$$\int_0^\infty d\omega\,\omega\,\mathrm{Im}\left[\frac{-1}{\epsilon(q,\omega)}\right] = \frac{\pi}{2}\,\omega_p^2.$$

(Mahan Ch. 5 contains a more complicated derivation of the f-sum rule that does not rely on any assumption about the large-ω behavior of the dielectric function.)

(g) Assuming that $\epsilon(q,\omega=0)$ diverges as $q\to 0$, as predicted by Thomas-Fermi theory, derive the sum rule

$$\lim_{q\to 0}\int_0^\infty \frac{d\omega}{\omega}\,\mathrm{Im}\left[\frac{-1}{\epsilon(q,\omega)}\right] = \frac{\pi}{2}.$$

Exact sum rules, such as those in (f) and (g) that involve moments of the loss function, can provide important tests for any approximate theory.

2. **Quasiparticle effective mass.** This question is designed to lead you through the derivation of the relation between the effective mass m^* of a translationally invariant Fermi liquid and its Landau parameters F_l^α, starting from the expansion

$$E = E_0 + \sum_{\mathbf{k},\sigma}\varepsilon^0_{\mathbf{k}\sigma}\,\delta n_{\mathbf{k}\sigma} + \frac{1}{2V}\sum_{\mathbf{k},\mathbf{k}'}\sum_{\sigma,\sigma'} f_{\mathbf{k}\sigma\mathbf{k}'\sigma'}\,\delta n_{\mathbf{k}\sigma}\,\delta n_{\mathbf{k}'\sigma'} + O(\delta n^3), \tag{2}$$

where $\varepsilon^0_{\mathbf{k}\sigma}$ is the energy of the quasiparticle state $(\mathbf{k},\sigma)$ in the ground state of the interacting Fermi system, and $\delta n_{\mathbf{k}\sigma} = n_{\mathbf{k}\sigma} - n^0_{\mathbf{k}\sigma}$ measures the change in the expected occupancy $n_{\mathbf{k}\sigma}$ from its ground-state value $n^0_{\mathbf{k}\sigma}$.

(a) Using Eq. (2), write down expressions for the quasiparticle energy $\varepsilon_{\mathbf{k}\sigma} = \partial E/\partial n_{\mathbf{k}\sigma}$ and for the quasiparticle velocity $\mathbf{v}_{\mathbf{k}\sigma} = \hbar^{-1}\boldsymbol{\nabla}_{\mathbf{k}}\,\varepsilon_{\mathbf{k}\sigma}$.

(b) Write down (i) the total current $\mathbf{I}_{qp}$ carried by all the quasiparticles in terms of $\mathbf{v}_{\mathbf{k}\sigma}$, the expected quasiparticle occupancies $n_{\mathbf{k}\sigma}$, and the quasiparticle charge $-e$; and (ii) the total momentum carried by the quasiparticles in terms of $\mathbf{k}$ and $n_{\mathbf{k}\sigma}$.

(c) Since the quasiparticles are in one-to-one correspondence with (and share the same wave vector and charge quantum numbers as) the true particles, it must be the case that $\mathbf{I}_{qp} = \mathbf{I}_p$ and $\mathbf{P}_{qp} = \mathbf{P}_p$, where $\mathbf{I}_p$ and $\mathbf{P}_p$ are the total current and total momentum calculated directly for the particles. Clearly $\mathbf{P}_p = -(m/e)\,\mathbf{I}_p$, and it therefore follows that $\mathbf{P}_{qp} = -(m/e)\,\mathbf{I}_{qp}$. Substitute your expressions for the quantities on each side of this last equation and then use your answer to (a) to eliminate $\mathbf{v}_{\mathbf{k}\sigma}$ and obtain an equation involving only $\hbar$, e, m, and quantities appearing in Eq. (2).

(d) Now calculate the change of each side of the equation you obtained in part (c) arising from an infinitesimal change $dn_{\mathbf{k}\sigma}$ in the occupancy $n_{\mathbf{k}\sigma}$ (and hence in $\delta n_{\mathbf{k}\sigma}$) for one particular state $(\mathbf{k},\sigma)$.

(e) By considering a case where the state $(\mathbf{k},\sigma)$ in part (d) lies on the Fermi surface, derive the relation

$$\frac{m^*}{m} = \frac{\hbar^2 k_F}{m}\Big/\left|\boldsymbol{\nabla}_{\mathbf{k}}\,\varepsilon^0_{\mathbf{k}\sigma}\right|_{|\mathbf{k}|=k_F} = 1 + \frac{F_1^s}{3}\,.$$

Hint: You may need to use integration by parts.