# PHZ 7427 Spring 2011 – Homework 3

Due by 5:00 p.m. on Wednesday, March 16. After that, the assignment may be submitted for 75% credit until the start of class on Monday, March 28.

*Answer all questions. Please write neatly and include your name on the front page of your answers. You must also clearly identify all your collaborators on this assignment. To gain maximum credit you should explain your reasoning and show all working.*

1. **Anisotropic Heisenberg ferromagnet** (based on Ashcroft and Mermin Problem 33–5). In insulators where the crystal field has lower-than-cubic symmetry, the nearest-neighbor spin Hamiltonian may take the form

$$\hat{H} = -\frac{1}{2}\sum_{\langle j,k\rangle}\left[J_z S_j^z S_k^z + \frac{1}{2}J_\perp\left(S_j^+ S_k^- + S_j^- S_k^+\right)\right],$$

where $J_z$ and $J_\perp$ are in general not equal.

   (a) Show that the state with all $S_j^z = S$ is an eigenstate of the Hamiltonian, whatever the values of $J_z$ and $J_\perp$.

   (b) Derive the magnon energies in linear spin-wave theory.

   (c) Consider the case $J_z > J_\perp > 0$. Show that the magnon spectrum is gapped. (This does not violate Goldstone's theorem, since the anisotropic Heisenberg model has discrete ($S^z \to -S^z$) rather than continuous spin rotation symmetry.) Find the low-temperature magnetization in $d$ spatial dimensions. Show that the deviation from saturation remains finite in dimensions $d \geq 2$ (in contrast to the situation discussed in class for $J_z = J_\perp$), and that under these conditions it is exponentially small in $-1/T$.

   (d) Consider the case $J_\perp > J_z > 0$. Show that some magnons have negative energies, which implies that the state from part (a) is not the ground state.

2. **Mean-field theory of ferrimagnetism and antiferromagnetism.** Ashcroft and Mermin Problem 33–7.

3. **Antiferromagnetic spin-wave theory.** This question leads you through the derivation of the standard linear spin-wave theory for a spin-$S$ nearest-neighbor Heisenberg antiferromagnet on a bipartite lattice, in which each site in sublattice $A$ has $z$ nearest neighbors in sublattice $B$, and vice versa. The spin Hamiltonian for this system is

$$\hat{H} = -J\sum_{j\in A}\sum_{\boldsymbol{\delta}} \mathbf{S}_j \cdot \mathbf{S}_{j+\boldsymbol{\delta}}$$

where $J < 0$ and site $j$ in sublattice $A$ has $z$ nearest neighbors $j + \boldsymbol{\delta}$ in sublattice $B$.

   (a) The system's classical ground state (the *Néel state*) has all spins in $A$ pointing along $\hat{z}$ (say), and all spins in $B$ pointing along $-\hat{z}$. Therefore, the natural vacuum state for spin-wave theory has $S^z = S$ in $A$ and $S^z = -S$ in $B$. However, it proves convenient to rewrite the problem using the original spin operators $\mathbf{S}_j$ for $j \in A$

and rotated spin operators $\tilde{\mathbf{S}}_k$ for $k \in B$, with $\tilde{S}_k^x = S_k^x$, $\tilde{S}_k^y = -S_k^y$, and $\tilde{S}_k^z = -S_k^z$. Verify that the operators $\tilde{\mathbf{S}}_k$ obey the standard spin commutation relations, and find the relation between $\tilde{S}_k^{\pm}$ and $S_k^{\pm}$. Show that the Hamiltonian can be rewritten

$$\hat{H} = -|J| \sum_{j \in A} \sum_{\boldsymbol{\delta}} \left[ S_j^z \tilde{S}_{j+\boldsymbol{\delta}}^z - \frac{1}{2} \left( S_j^+ \tilde{S}_{j+\boldsymbol{\delta}}^+ + S_j^- \tilde{S}_{j+\boldsymbol{\delta}}^- \right) \right].$$

(b) Perform a Holstein-Primakoff transformation on the spins $\mathbf{S}_j$ and $\tilde{\mathbf{S}}_k$ in terms of bosonic operators $a_j$ and $b_k$, respectively. Show that the magnon (quadratic) part of the resulting Hamiltonian can be written

$$\hat{H}_{\text{magnon}} = |J| S z \sum_{\mathbf{k}} \left[ a_{\mathbf{k}}^\dagger a_{\mathbf{k}} + b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \gamma_{\mathbf{k}} \left( a_{\mathbf{k}}^\dagger b_{-\mathbf{k}}^\dagger + a_{\mathbf{k}} b_{-\mathbf{k}} \right) \right],$$

where $\gamma_{\mathbf{k}} = z^{-1} \sum_{\boldsymbol{\delta}} \exp(i\mathbf{k} \cdot \boldsymbol{\delta})$. Here $\mathbf{k}$ runs over the first Brillouin zone of the *magnetic lattice*, which is smaller than the crystalline Brillouin zone since the real-space unit volume of each sublattice is twice that of the full lattice. (*Interacting Electrons and Quantum Magnetism* by A. Auerbach treats this problem instead using a single set of bosonic operators $b_{\mathbf{k}}$ with $\mathbf{k}$ extending over the first Brillouin zone of the crystal lattice.)

(c) Perform a *Bogoliubov transformation*

$$a_{\mathbf{k}} = \cosh\theta_{\mathbf{k}} \alpha_{\mathbf{k}} + \sinh\theta_{\mathbf{k}} \beta_{-\mathbf{k}}^\dagger, \qquad b_{\mathbf{k}} = \cosh\theta_{\mathbf{k}} \beta_{\mathbf{k}} + \sinh\theta_{\mathbf{k}} \alpha_{-\mathbf{k}}^\dagger$$

with the real parameters $\theta_{\mathbf{k}} \equiv \theta_{-\mathbf{k}}$ chosen to eliminate any "anomalous" terms of the form $\alpha^\dagger \beta^\dagger$ or $\alpha\beta$. Verify that

$$\alpha_{\mathbf{k}} = \cosh\theta_{\mathbf{k}} a_{\mathbf{k}} - \sinh\theta_{\mathbf{k}} b_{-\mathbf{k}}^\dagger, \qquad \beta_{\mathbf{k}} = \cosh\theta_{\mathbf{k}} b_{\mathbf{k}} - \sinh\theta_{\mathbf{k}} a_{-\mathbf{k}}^\dagger$$

obey bosonic commutation relations.

(d) Find the dispersion relation $\omega_{\mathbf{k}}$ of the resulting magnon modes. Show that there are two spin waves with vanishing $\omega_{\mathbf{k}}$, and find the dispersion near the zeros.

(e) The ground state of the antiferromagnet is the state with no $\alpha$ or $\beta$ bosons. However, this does not correspond to a state with no $a$ or $b$ bosons. In fact, the quantum-mechanical ground state contains arbitrary numbers of spins having $|S^z| < S$. Show that the ground-state energy predicted by linear spin-wave theory is lower than that of our originally assumed vacuum state.

(f) The order parameter that assumes a nonzero value in the antiferromagnetic phase is the staggered magnetization (per unit volume)

$$M_s = M_A - M_B, \qquad M_A = \frac{g\mu_B}{V} \left\langle \sum_{j \in A} S_j^z, \right\rangle \qquad M_B = \frac{g\mu_B}{V} \left\langle \sum_{k \in B} S_k^z \right\rangle.$$

Calculate $M_s(T)$ in the low-temperature limit, commenting on whether there seems to be a failure of the spin-wave theory for any particular value(s) of the spatial dimensionality $d$.