Summary of Complex Derivatives and Contour Integrals

- The derivative of a complex function $f(z)$ is

$$\frac{df}{dz} = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}. \tag{1}$$

- Let $z = x + iy$, and $f(z) = u(x, y) + iv(x, y)$, where u and v are real-valued. For $dz = dx$, the derivative of f is

$$\frac{df}{dz} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial z} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \tag{2}$$

while for $dz = idy$,

$$\frac{df}{dz} = \frac{\partial f}{\partial y}\frac{\partial y}{\partial z} = -i\left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right). \tag{3}$$

These two derivatives are equal if $f(z)$ satisfies the *Cauchy-Riemann* conditions

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \text{and} \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}. \tag{4}$$

Equations (4) imply that $\nabla^2 u = \nabla^2 v = 0$.

- A complex function $f(z)$ is *analytic* (or *complex differentiable*) at $z = z_0$ if df/dz defined by Eq. (1) exists and its value is independent of the direction of dz on the complex plane. It turns out that the second requirement is met if $f(z)$ has continuous partial derivatives and satisfies the Cauchy-Riemann conditions.

- A *singularity* of a complex function $f(z)$ is a point where $f(z)$ is nonanalytic. If $(z - z_0)^n f(z)$ is not analytic for any finite positive integer n, then the point $z = z_0$ is an *essential singularity*. The singularity is a *pole* of order p if p is the smallest positive integer for which $(z - z_0)^p f(z)$ is analytic at $z = z_0$.

- A complex function $f(z)$ is *analytic* (or *holomorphic*) on a region R if it is differentiable at every point in R. Such a function is infinitely differentiable at every point in R.

- A complex function $f(z)$ is *entire* if it is analytic at all finite z. Examples include all polynomials $\sum_{j=0}^{n} a_j z^j$, $\exp(z)$, $\cos(z)$, $\sin(z)$, $Ai(z)$, $Bi(z)$, $1/\Gamma(z)$, and $\mathrm{erf}(z)$.

- A *meromorphic* function is a complex function of the form $f(z) = g(z)/h(z)$, where $g(z)$ and $h(z)$ are entire and $h(z)$ is not identically zero. Such a function is analytic, except possibly at isolated poles.

- A *contour integral* is an integral obtained by integration of a complex function along a prescribed path (the contour) in the complex plane. The integral of $f(z)$ around a closed contour C is denoted $\oint_C f(z)\, dz$. By convention, closed contours are traversed in a counter-clockwise direction.

- *Cauchy integral theorem*: If $f(z)$ is analytic on a simply connected region R, then

$$\oint_C f(z)\,dz = 0 \tag{5}$$

for any closed contour C completely contained in R. If, instead, $f(z)$ is analytic on a multiply connected region R, then

$$\oint_C f(z)\,dz = \sum_j \oint_{C_j} f(z)\,dz \tag{6}$$

where the sum runs over all the nonanalytic regions R_j enclosed by contour C, and C_j is any closed contour completely within R that encloses R_j.

- *Cauchy integral formula*: If $f(z)$ is analytic on a simply connected region R, then

$$f(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)\,dz}{z - z_0}, \tag{7}$$

for any closed contour C completely contained in R that encloses the point z_0.

A corollary of Eq. (7) is that

$$\left.\frac{d^n f}{dz^n}\right|_{z_0} = \frac{n!}{2\pi i}\oint_C \frac{f(z)\,dz}{(z - z_0)^{n+1}}. \tag{8}$$

- *Laurent series*: Suppose that a complex function $f(z)$ is analytic in a region R enclosing but not necessarily including $z = z_0$ (e.g., an annulus centered on z_0). Then for any point z in R,

$$f(z) = \sum_{n=-\infty}^{\infty} a_n(z_0)(z - z_0)^n, \qquad a_n(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)\,dz}{(z - z_0)^{n+1}}, \tag{9}$$

where C is a closed contour completely contained in R that encloses the point z_0. If $f(z)$ is analytic at $z = z_0$, then $a_n(z_0) = 0$ for all $n < 0$, and the Laurent series reduces to the Taylor-McLaurin series. If z_0 is a pole of order p, then $a_{-p}(z_0) \neq 0$, but $a_n(z_0) = 0$ for all $n < -p$.

- *Residue theorem*: If a complex function $f(z)$ is analytic apart from isolated poles on some region R, then for any contour C completely contained in R

$$\oint_C f(z)\,dz = 2\pi i \sum_j a_{-1}(z_j), \tag{10}$$

where the sum runs over all poles z_j enclosed by the contour. The Laurent series coefficient $a_{-1}(z_j)$ is the *residue* of $f(z)$ at $z = z_j$.