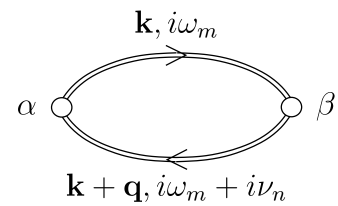
PHZ 7428 Fall 2004 – Homework 2

Due by 4:05 p.m. on Wednesday, November 10.

Answer both questions. Please write neatly and include your name on the front page of your answers. You must also clearly identify all your collaborators on this assignment. To gain maximum credit you should explain your reasoning and show all working.

1. *Frequency dependence of the Drude conductivity.* In class, we calculated the leading contribution to the electrical conductivity in the limit $\mathbf{q} \to \mathbf{0}$, then $\omega \to 0$. The aim of this question is to calculate the small-$|\mathbf{q}|$, small-$|\omega|$ wavevector and frequency dependence of our lowest-order conductivity approximation $\sigma^{(0)}_{\alpha,\beta}$, derived from

$$\hbar\bar{\Pi}^{(0)}_{\alpha,\beta}(\mathbf{q}, i\nu_n) \;=$$



Take as your starting point the expression we found in class:

$$\hbar\bar{\Pi}^{(0)}_{\alpha,\beta}(\mathbf{q}, \omega + i\eta) \;=\; \frac{\hbar^2 q^2}{2m^2}\int \frac{d^3\mathbf{k}}{(2\pi)^3}\,(2k_\alpha + q_\alpha)(2k_\beta + q_\beta)\int_{-\infty}^{\infty} dx\, n_F(\hbar x)\times$$
$$\times\left[\bar{G}^R(\mathbf{k}+\mathbf{q}, x+\omega)\bar{A}(\mathbf{k}, x) + \bar{G}^A(\mathbf{k}, x-\omega)\bar{A}(\mathbf{k}+\mathbf{q}, x)\right],$$

where

$$\bar{G}^{R}_{A}(\mathbf{k}, \omega) \approx \frac{1}{\omega - \omega_{\mathbf{k}} \pm i/2\tau}.$$

In order to obtain the leading behavior of $\bar{\Pi}^{(0)}_{\alpha,\beta}(\mathbf{q}, \omega + i\eta)$, we make the following additional approximations:

(i) Set $2k_\alpha + q_\alpha \approx 2k_F\hat{k}_\alpha$, and $2k_\beta + q_\beta \approx 2k_F\hat{k}_\beta$. Here $\hat{k}_\alpha$ is the α component of the unit vector $\hat{\mathbf{k}}$ that points in the same direction as $\mathbf{k}$.

(ii) Set $\omega_{\mathbf{k}+\mathbf{q}} = \omega_{\mathbf{k}} + \mathbf{v}(\mathbf{k})\cdot\mathbf{q} + \hbar|\mathbf{q}|^2/2m \approx \omega_{\mathbf{k}} + v_F\,\hat{\mathbf{k}}\cdot\mathbf{q}$.

(iii) Set
$$\int_0^\infty \frac{k^2 dk}{(2\pi)^3}\, f(\omega_k) = \int_{-\infty}^{\infty}\frac{N(\hbar y)\hbar dy}{4\pi} f(y) \approx \frac{\hbar N(0)}{4\pi}\int_{-\infty}^{\infty} dy\, f(y).$$

(a) Show that, as a result of approximations (i)–(iii), y enters the integrand only in the combination $x - y$, so

$$\bar{\Pi}^{(0)}_{\alpha,\beta}(\mathbf{q}, \omega + i\eta) = \frac{\hbar^2 q^2 k_F^2 N(0)}{2\pi m^2}\int d^2\hat{\mathbf{k}}\,\hat{k}_\alpha\hat{k}_\beta\int_{-\infty}^{\infty} dy\int_{-\infty}^{\infty} dx\, n_F(\hbar x)F(x-y),$$

where $d^2\hat{\mathbf{k}} = \sin\theta_k \, d\theta_k \, d\phi_k$. Taking into account the form of $F(x-y)$, show that it is legitimate to perform the following sequence of manipulations:

$$\begin{aligned}
\int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dx \, n_F(\hbar x) F(x-y) &= -\int_{-\infty}^{\infty} dy \, y \, \frac{d}{dy} \int_{-\infty}^{\infty} dx \, n_F(\hbar x) F(x-y) \\
&= \int_{-\infty}^{\infty} dy \, y \int_{-\infty}^{\infty} dx \, n_F(\hbar x) \frac{dF(x-y)}{dx} \\
&= -\hbar \int_{-\infty}^{\infty} dy \, y \int_{-\infty}^{\infty} dx \, n_F'(\hbar x) F(x-y) \\
&= -\hbar \int_{-\infty}^{\infty} dx \, n_F'(\hbar x) \int_{-\infty}^{\infty} dy \, y F(x-y),
\end{aligned}$$

where $n_F'(\epsilon) = dn_F(\epsilon)/d\epsilon$. The endpoint of this sequence turns out to greatly simplify the evaluation of the integrals over x and y.

(b) Show that

$$\int_{-\infty}^{\infty} dy \, y F(x-y) = \frac{\omega}{\omega - v_F \hat{\mathbf{k}} \cdot \mathbf{q} + i/\tau} - 1. \tag{1}$$

(c) Carry out the integration over x, and hence obtain an expression for $\bar{\Pi}^{(0)}_{\alpha,\beta}(\mathbf{q}, \omega + i\eta)$ as an integral over $\hat{\mathbf{k}}$. Eliminate k_F and $N(0)$ from your expression in favor of the electron density n.

(d) Use your result from the previous part to show that

$$\sigma^{(0)}_{\alpha,\beta}(\mathbf{q}, \omega) = \frac{3nq^2\tau}{4\pi m} \int d^2\hat{\mathbf{k}} \, \frac{\hat{k}_\alpha \hat{k}_\beta}{1 - i\omega\tau + il \, \hat{\mathbf{k}} \cdot \mathbf{q}},$$

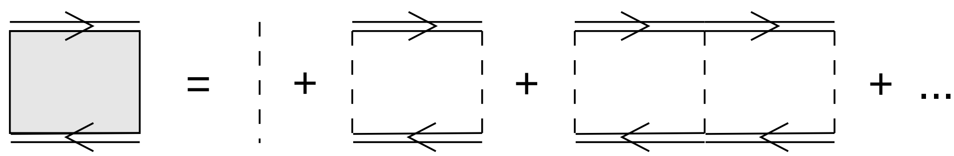
where $l = v_F \tau$ is the mean-free path. Simplify this result for the special case $\mathbf{q} = \mathbf{0}$.

(e) Focus on $\mathbf{q} = \mathbf{0}$, in which case only approximations (i) and (iii) are applicable. Show that retaining the full expression $4k_\alpha k_\beta$ in the current vertices and expanding $N(\hbar\omega) \approx N(0) + \hbar\omega N'(0)$ does not change the calculated conductivity.

2. *Calculation of the diffuson.* In class we calculated the cooperon in d spatial dimensions for a zero-range impurity

$$V_i(\mathbf{r}) = V_0 a^d \delta^d(\mathbf{r}).$$

In this question, you will calculate the diffuson (the diffusion propagator)



for the same potential.

(a) Starting from the integral equation

$$\begin{aligned} D(\mathbf{k},\mathbf{k}',\mathbf{q},i\omega_m,i\nu_n) &= \frac{n_i}{\hbar^2}|V_i(\mathbf{k}'-\mathbf{k})|^2 + \frac{n_i}{\hbar^2}\int\frac{d^d\mathbf{k}_1}{(2\pi)^d}|V_i(\mathbf{k_1}-\mathbf{k})|^2\bar{G}(\mathbf{k}_1,i\omega_m)\times \\ &\qquad \times\bar{G}(\mathbf{k}_1+\mathbf{q},i\omega_m+i\nu_n)D(\mathbf{k}_1,\mathbf{k}',\mathbf{q},i\omega_m,i\nu_n), \end{aligned}$$

show that for the zero-range potential,

$$D(\mathbf{k},\mathbf{k}',\mathbf{q},i\omega_m,i\nu_n) = D(\mathbf{q},i\omega_m,i\nu_n) = \frac{v^2}{1-v^2 I(\mathbf{q},i\omega_m,i\nu_n)},$$

where $v^2 = n_i(V_0 a^d)^2/\hbar^2$ and

$$I(\mathbf{k},\mathbf{k}',\mathbf{q},i\omega_m,i\nu_n) = \int\frac{d^d\mathbf{k}}{(2\pi)^d}\bar{G}(\mathbf{k},i\omega_m)\bar{G}(\mathbf{k}+\mathbf{q},i\omega_m+i\nu_n).$$

[Comparison with the lecture notes should reveal that for this potential, the diffuson and the cooperon are related via $D(\mathbf{q},i\omega_m,i\nu_n) = C(-\mathbf{q},i\omega_m,i\nu_n)$.]

(b) Set

$$\bar{G}(\mathbf{k},i\omega_m) \approx \frac{1}{i\omega_m - \omega_\mathbf{k} + (i/2\tau)\,\mathrm{sgn}\omega_m},$$

$\omega_{\mathbf{k}+\mathbf{q}} \approx \omega_\mathbf{k} + |\mathbf{q}|v_F\cos\theta$, where θ is the angle between $\mathbf{k}$ and $\mathbf{q}$, and $N(\hbar\omega_\mathbf{k}) \approx N(0)$, all of which are good approximations for $\hbar|\omega_\mathbf{k}| \ll \epsilon_F$ and $|\mathbf{q}| \ll k_F$. Hence perform the integral over the magnitude of $\mathbf{k}$.

(c) Perform the integral over the direction of $\mathbf{k}$ for each of the cases (i) $d=3$, (ii) $d=2$, and (iii) $d=1$. Do not rely on the assumption that $|\nu_n|\tau \ll 1$ or $|\mathbf{q}|l \equiv |\mathbf{q}|v_F\tau \ll 1$. The integral is straightforward for $d=3$ and $d=1$.
For $d=2$, show by writing $z=e^{i\theta}$ that

$$\int_0^{2\pi}\frac{d\theta}{w+\cos\theta} = -2i\oint_C\frac{dz}{1+2wz+z^2},$$

where w is a complex number and the contour C runs in a counterclockwise direction around the circle $|z|=1$. Evaluate the integral using the residue theorem.

(d) Use the result of the preceding part to obtain an expression for the diffuson that is valid not only in the *diffusive limit* $|\omega|\tau \ll 1$, $|\mathbf{q}|l \ll 1$ but also in the *ballistic limit* $|\omega|\tau \gg 1$, $|\mathbf{q}|l \gg 1$ (provided that $\hbar|\omega| \ll \epsilon_F$ and $|\mathbf{q}| \ll k_F$).