PHZ 7428 Fall 2004 – Homework 3

Due by 4:05 p.m. on Friday, December 3.

Answer both questions. Please write neatly and include your name on the front page of your answers. You must also clearly identify all your collaborators on this assignment. To gain maximum credit you should explain your reasoning and show all working.

1. *Retarded polarization function of a noninteracting electron gas.* In class, we discussed the polarization function of the noninteracting three-dimensional electron gas:

$$\Pi_0(\mathbf{q}, z) = 2 \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{n_F(\epsilon_\mathbf{k}) - n_F(\epsilon_{\mathbf{k}+\mathbf{q}})}{\hbar z + \epsilon_\mathbf{k} - \epsilon_{\mathbf{k}+\mathbf{q}}}$$

   (a) At zero temperature, Mahan (Sec. 5.1) evaluates $\Pi_0(\mathbf{q}, i\nu_n)$, while Fetter and Walecka (Sec. 12) show in detail how to evaluate the time-ordered polarization function $\Pi_0^T(\mathbf{q}, \omega)$.

   Using one or other of these starting points, find the $T = 0$ retarded polarization function $\Pi_0^R(\mathbf{q}, \omega)$. Make sure that your answer covers both positive and negative ω.

   (b) Use your result from (a) to calculate the dielectric constant $\epsilon(\mathbf{q}, \omega)$ in the RPA approximation for two limits: (i) $\mathbf{q}$ fixed, $\omega \to 0$; (ii) ω fixed, $\mathbf{q} \to 0$.

2. *RPA for a two-dimensional electron gas.* An effective two-dimensional electron gas can be created at the planar interface between two semiconductors (e.g., GaAs and GaAlAs). We will idealize such a device using a jellium model of electrons having a density (per unit area) n and an effective mass m^*. It is important to note that although the electrons are confined to move in two dimensions, they interact via the *three-dimensional* Coulomb potential

$$V(\mathbf{r}) = \frac{e^2}{\kappa|\mathbf{r}|},$$

   where κ depends on the static dielectric constants of the semiconductors on either side of the interface.

   (a) Show that at zero temperature, the noninteracting retarded polarization function

$$\Pi_0^R(\mathbf{q}, \omega) = 2 \int \frac{d^2\mathbf{k}}{(2\pi)^2} \frac{n_F(\epsilon_\mathbf{k}) - n_F(\epsilon_{\mathbf{k}+\mathbf{q}})}{\hbar\omega + \epsilon_\mathbf{k} - \epsilon_{\mathbf{k}+\mathbf{q}} + i0^+},$$

   where $\mathbf{q}$ is constrained to lie in the plane of the electron gas, takes the form

$$\begin{aligned} \Pi_0^R(\mathbf{q}, \omega) &= \frac{m^*}{2\pi\hbar^2 x} \left[S(x_+^2 - 1) \operatorname{sgn} x_+ + S(x_-^2 - 1) \operatorname{sgn} x_- - 2x \right. \\ &\quad \left. + iS(1 - x_+^2) - iS(1 - x_-^2) \right], \end{aligned}$$

where $x = |\mathbf{q}|/2k_F$, $x_\pm = x \pm \omega/v_F|\mathbf{q}|$, and

$$S(y) = \begin{cases} 0 & \text{for } y < 0, \\ \sqrt{y} & \text{for } y \geq 0. \end{cases}$$

You may find it helpful to use the formula introduced in Homework 2:

$$\int_0^{2\pi} \frac{d\theta}{w + \cos\theta} = -2i \oint_C \frac{dz}{1 + 2wz + z^2},$$

where w is a complex number and the contour C runs in a counterclockwise direction around the circle $|z| = 1$.

(b) Calculate the $T = 0$ dielectric response function $\epsilon(\mathbf{q}, \omega)$ within the RPA approximation. You may assume that the RPA dielectric function is given by $\epsilon(\mathbf{q}, \omega) = \kappa[1 + V(\mathbf{q})\Pi_0(\mathbf{q}, \omega)]$, where $V(\mathbf{q})$ is the *two-dimensional* Fourier transform of the three-dimensional Coulomb potential: $V(\mathbf{q}) = 2\pi e^2/\kappa|\mathbf{q}|$.

(c) Calculate the plasmon dispersion relation $\omega(q)$. Over what range of wavenumbers q are the plasmons strongly damped?

(d) Working in the RPA approximation, calculate the screened Coulomb interaction $V_{\mathrm{RPA}}(\mathrm{q}, \omega)$.

(e) Show that in the low-frequency, long-wavelength limit, the screened Coulomb interaction reduces to the form

$$V_{\mathrm{RPA}}(\mathbf{q}) = \frac{2\pi e^2}{\kappa(Q + |\mathbf{q}|)}.$$

Show that the screening wavevector Q is independent of the electron density.

(f) Calculate Q/k_F for a two-dimensional electron gas in which $m^* = 0.1m$, $\kappa = 10$, and $n = 10^{15}\,\mathrm{m}^{-2}$ (all fairly realistic values for a semiconductor heterostructure).