

# SELECTED TOPICS

IN

## QUANTUM THEORY

1. COHERENT STATES
2. PATH INTEGRALS
3. QUANTUM CONSTRAINTS
4. QUANTUM GRAVITY

by

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# COHERENT STATES

First and foremost used for representations

Analogs

Discrete basis:

$$\{|n\rangle\}_{n=0}^{\infty}, \sum_{n=0}^{\infty} |n\rangle \langle n| = I, \langle n|m\rangle = \delta_{nm}$$

$$|\psi\rangle \in \mathcal{H} \rightarrow \langle n|\psi\rangle = \psi_n \in \ell^2$$

$$\langle \varphi|\psi\rangle = \langle \varphi|I|\psi\rangle = \sum_n \langle \varphi|n\rangle \langle n|\psi\rangle = \sum_n \varphi^*_n \psi_n$$

$$B \in \mathcal{O} \rightarrow \langle n|B|m\rangle = B_{nm}; N|n\rangle = n|n\rangle$$

Continuous basis:

$$\{|x\rangle\}_{x \in \mathbb{R}}, \int |x\rangle \langle x| dx = I, \langle x|\psi\rangle = \delta(x-y)$$

$$|\psi\rangle \in \mathcal{H} \rightarrow \langle x|\psi\rangle = \psi(x) \in L^2(\mathbb{R})$$

$$\langle \varphi|\psi\rangle = \int \langle \varphi|x\rangle \langle x|\psi\rangle dx = \int \varphi(x)^* \psi(x) dx$$

$$B \in \mathcal{O} \rightarrow \langle x|B|y\rangle = B(x, y); Q|x\rangle = x|x\rangle$$

Canonical coherent states:

$$\{|z\rangle\}_{z \in \mathbb{C}}, \int |z\rangle \langle z| d\mu(z) = I,$$

$$\langle z|z'\rangle = e^{-|z|^2/2 + z^* z' - |z'|^2/2}$$

$$d\mu(z) = \frac{d\text{Re}(z)}{\pi} \frac{d\text{Im}(z)}{\pi} = \frac{dx dy}{\pi}$$

$$|\psi\rangle \in \mathcal{H} \rightarrow \langle z|\psi\rangle = \psi(z) \in L^2(\mathbb{R}^2)$$

$$\langle \varphi|\psi\rangle = \int \langle \varphi|z\rangle \langle z|\psi\rangle d\mu = \int \varphi(z)^* \psi(z) d\mu$$

$$B \in \mathcal{O} \rightarrow \langle z|B|z'\rangle; a|z\rangle = z|z\rangle$$

$$[a, a^\dagger] = aa^\dagger - a^\dagger a = I ; N = a^\dagger a$$

$$N|n\rangle = n|n\rangle , n=0, 1, 2, \dots$$

$$a|n\rangle = \sqrt{n}|n-1\rangle , a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$$

$$|z\rangle = e^{-|z|^2/2} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} |n\rangle ; a|z\rangle = z|z\rangle$$

$$\langle z| = e^{-|z|^2/2} \sum_{n=0}^{\infty} \frac{z^{*n}}{\sqrt{n!}} \langle n| ; \langle z| a^\dagger = z^* \langle z|$$

Resolution of unity

$$z = n e^{i\theta}$$

$$\int |z\rangle \langle z| \frac{dx dy}{\pi} = \sum_{n,m} \frac{|n\rangle \langle m|}{\sqrt{n!m!}} \int e^{-|z|^2} z^n z^{*m} dx dy / \pi$$

$$= \sum_n \frac{|n\rangle \langle n|}{n!} \int e^{-r^2} r^{2n} 2\pi r dr = \sum_n |n\rangle \langle n| = I$$

Normal ordering : Use  $a a^\dagger = I + a^\dagger a$

$$B(a^\dagger, a) = :C(a^\dagger, a):$$

$$\begin{aligned} \langle z| B(a^\dagger, a) |z'\rangle &= \langle z| :C(a^\dagger, a)|z'\rangle \\ &= C(z^*, z') \langle z| z'\rangle = B(z, z') \end{aligned}$$

Diagonal coherent state matrix elements

$$\begin{aligned} \langle z| \sum C_{mn} a^{*m} a^n |z\rangle &= \sum C_{mn} z^{*m} z^n \\ &= \sum C_{mn} r^{m+n} e^{i(n-m)\theta} \end{aligned}$$

$$\langle z| B |z'\rangle \iff \langle z| B |z\rangle = B(z)$$

Diagonal representation of operators

$$\begin{aligned} \sum D_{mn} a^m a^{*n} &= \sum D_{mn} \int a^m |z\rangle \langle z| a^{*n} d\mu \\ &= \int (\sum D_{mn} z^m z^{*n}) |z\rangle \langle z| d\mu \end{aligned}$$

(anti) normal-ordered symbols

Bargmann space (Segal-Bargmann)

$|\psi\rangle \rightarrow f(z)$  entire function

$$\langle\varphi|\psi\rangle \rightarrow \int g(z)^* f(z) e^{-|z|^2} dx dy / \pi$$

$$a \rightarrow \frac{d}{dz}, \quad a^\dagger \rightarrow z \quad \boxed{\text{WHY}}$$

Harmonic oscillator Hamiltonian

$$H = \hbar\omega(a^\dagger a + \frac{1}{2}) \rightarrow \hbar\omega(z \frac{d}{dz} + \frac{1}{2})$$

$$\hbar\omega(z \frac{d}{dz} + \frac{1}{2}) f_n(z) = E_n f_n(z)$$

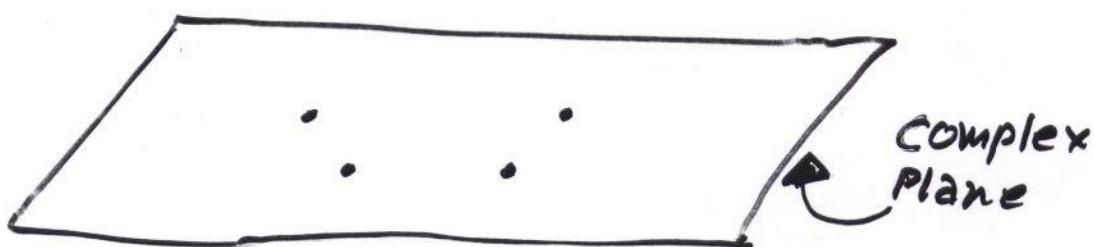
$$z \frac{d}{dz} f_n = \left( \frac{E_n}{\hbar\omega} - \frac{1}{2} \right) f_n$$

$$f_n \propto z^{\left(\frac{E_n}{\hbar\omega} - \frac{1}{2}\right)} = z^n, \quad n=0,1,2,\dots$$

$$\therefore E_n = \hbar\omega(n + \frac{1}{2})$$

Representation of  $f(z)$  by its zeros

$$f(z) = (z - z_1)(z - z_2) \dots (z - z_p)$$



A few complex numbers specify  $f(z)$

## Connection with Phase Space

$$|\beta\rangle = e^{(\beta a^\dagger - \beta^* a)} |0\rangle$$

$$= e^{-\frac{1}{2}|\beta|^2} e^{\beta a^\dagger} e^{-\beta a} |0\rangle$$

**WHY**

$$= e^{-\frac{1}{2}|\beta|^2} e^{\beta a^\dagger} |0\rangle, a|0\rangle = 0$$

$$= e^{-\frac{1}{2}|\beta|^2} \sum_{n=0}^{\infty} \frac{\beta^n}{n!} a^{+n} |0\rangle$$

$$= e^{-\frac{1}{2}|\beta|^2} \sum_{n=0}^{\infty} \frac{\beta^n}{\sqrt{n!}} |n\rangle$$

.....

$$\beta = \frac{g + iP}{\sqrt{2\pi}}, \quad a = \frac{Q + iP}{\sqrt{2\pi}}$$

$$e^{\frac{i}{\hbar}[(g+iP)(Q-iP) - (g-iP)(Q+iP)]}$$

$$= e^{\frac{i}{\hbar}(PQ - gP)} = e^{(za^\dagger - z^*a)}$$

Weyl operator

Generalized canonical coherent states

$$\{ |p, g; \eta\rangle \equiv e^{i/\hbar(PQ - gP)} |\eta\rangle = |p, g\rangle$$

$$\{ |p, g; n\rangle \equiv e^{-i/\hbar gP} e^{i/\hbar PQ} |\eta\rangle = |p, g\rangle$$

differ by a phase

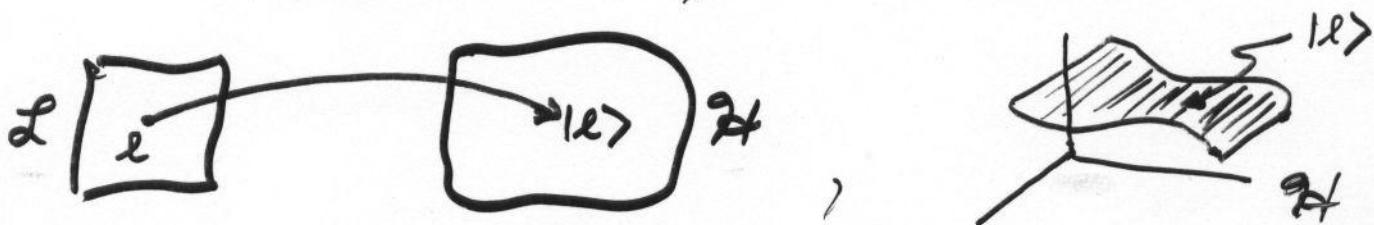
**WHAT**

Resolution of unity

$$\int |p, g\rangle \langle p, g| \frac{dp dq}{2\pi\hbar} = I = I \langle \eta | \eta \rangle$$

# General definition of coherent states

$$\ell \in \mathcal{L} \approx \mathbb{R}^n, \quad |\ell\rangle \in \mathcal{H}$$



1) continuous map:  $\ell' \rightarrow \ell \rightarrow |\ell'\rangle \rightarrow |\ell\rangle$   
 i.e.,  $\langle \ell' | \ell \rangle$  jointly cont., or  $\| |\ell'\rangle - |\ell\rangle \| \rightarrow 0$

2) positive measure:  $\mu(\ell)$ ,  $d\mu(\ell) = P(\ell) d^n \ell$

$$\int |\ell\rangle \langle \ell| d\mu(\ell) = I$$

Remarks:

- i)  $|\ell\rangle \neq 0, P(\ell) > 0$  (a.e.)
- ii) can normalize so  $\langle \ell | \ell \rangle = \| |\ell\rangle \|^2 = 1$
- iii) groups are useful but not required

## Group examples:

• Rotation group (spin  $s$ ,  $s=\frac{1}{2}, 1, \frac{3}{2}, \dots$ )

$$[S_j, S_k] = i \epsilon_{jkl} S_l; \quad \sum_s S_s^2 = s(s+1) I$$

$$|n\rangle = |s\rangle, \quad S_3 |s\rangle = s |s\rangle$$

$$|\theta, \varphi\rangle \equiv e^{-i\varphi S_3} e^{-i\theta S_2} |s\rangle, \quad \| |\theta, \varphi\rangle \| = 1$$

$$I = \int |\theta, \varphi\rangle \langle \theta, \varphi| d\mu(\theta, \varphi)$$

$$= \frac{2s+1}{4\pi} \int |\theta, \varphi\rangle \langle \theta, \varphi| \sin \theta d\theta d\varphi$$

WHY

$$|\psi\rangle \in \mathcal{H} \rightarrow \langle \theta, \varphi | \psi \rangle = \psi(\theta, \varphi) \in L^2(S^2)$$

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$$B \in \mathcal{O} \rightarrow \langle \theta, \varphi | B | \theta', \varphi' \rangle$$

$$\langle \theta, \varphi | B | \theta', \varphi' \rangle \iff \langle \theta, \varphi | B | \theta, \varphi \rangle = B(\theta, \varphi)$$

$$B = \int b(\theta, \varphi) | \theta, \varphi \rangle \langle \theta, \varphi | d\mu(\theta, \varphi)$$

$$\langle \theta, \varphi | B | \theta', \varphi' \rangle = \left[ \cos \frac{\theta}{2} \cos \frac{\varphi}{2} e^{-i \frac{1}{2} (\varphi - \varphi')} + \sin \frac{\theta}{2} \sin \frac{\varphi}{2} e^{i \frac{1}{2} (\varphi - \varphi')} \right]^2$$

• Affine group (wavelets)

$$[G, P] = i\hbar I, \quad Q > 0$$

$$e^{i\theta P/\hbar} Q e^{-i\theta P/\hbar} = Q + \theta, \quad Q > 0 \text{ violated } \frown \text{ smiley}$$

$\therefore P$  is not observable

$$[G, P] \cdot Q = i\hbar Q = [Q, D], \quad D = \frac{1}{2}(PQ + QP)$$

$$e^{i\theta D/\hbar} Q e^{-i\theta D/\hbar} = e^\theta Q, \quad Q > 0 \text{ preserved } \smile$$

$$\text{choose } [Q, D] = i\hbar G, \quad G > 0$$

$$\langle n | Q | n \rangle = 1, \quad \langle n | D | n \rangle = 0$$

$$[Q - 1 + i\beta^{-1}\hbar^{-1}D] | n \rangle = 0$$

$$| p, g \rangle = e^{iPQ/\hbar} e^{-i\hbar \ln g D} | n \rangle$$

$$I = \int | p, g \rangle \langle p, g | d\mu(p, g)$$

$$d\mu(p, g) = \frac{dp dg}{2\pi\hbar C}, \quad C = \langle n | Q^{-1} | n \rangle < \infty$$

$(\beta > \frac{1}{2})$

$$|\psi\rangle \in \mathcal{H} \rightarrow \langle p, g | \psi \rangle = \psi(p, g) \in L^2(\mathbb{R} \times \mathbb{R}^+)$$

$$B \in \mathcal{O} \rightarrow \langle p, g | B | p', g' \rangle$$

$$\langle p, g | B | p', g' \rangle \Leftrightarrow \langle p, g | B | p, g \rangle = B(p, g)$$

$$B = \int b(p, g) |p, g\rangle \langle p, g| d\mu(p, g)$$

$$\langle p, g | p', g' \rangle = \left\{ \frac{g^{-k_2} g'^{-k'_2}}{\left[ \frac{1}{2}(g^{-1} + g'^{-1}) + \frac{c}{2\beta\pi} (p - p') \right]} \right\}^{2\beta}$$

Non-group related example

$$|\beta\rangle = N \sum_{n=0}^{\infty} \frac{\beta^n}{\sqrt{P_n}} |n\rangle, \quad N^{-2} = \sum_{n=0}^{\infty} \frac{| \beta |^{2n}}{P_n}, \quad |\beta| < U$$

$$I = \int_{|\beta| < U} |\beta\rangle \langle \beta| d\mu(\beta), \quad d\mu = W(|\beta|) \frac{dx dy}{\pi}$$

$$\begin{aligned} &= \sum_{n=0}^{\infty} \frac{|n\rangle \langle n|}{P_n} \int N^2(|\beta|) W(|\beta|) |\beta|^{2n} d|\beta|^2 \\ &= \sum_{n=0}^{\infty} |n\rangle \langle n| = I \end{aligned}$$

Example:

$$N^2 W = \frac{1}{2} e^{-|\beta|}, \quad P_n = (2n+1)!$$

$$|\beta\rangle = \sqrt{\frac{1}{\sinh(|\beta|)}} \sum_{n=0}^{\infty} \frac{\beta^n}{\sqrt{(2n+1)!}} |n\rangle, \quad U = \infty$$

$$I = \int |\beta\rangle \langle \beta| \left\{ \frac{(1 - e^{-2|\beta|})}{4|\beta|} \right\} \frac{dx dy}{\pi} \quad \boxed{\text{CHECK}}$$

# Physics of coherent states: Classical & quantum mechanics

$$Q|x\rangle = x|x\rangle, \quad \dim[x] = \dim[Q]$$

$$P|k\rangle = k|k\rangle, \quad \dim[k] = \dim[P]$$

$$U[p, q] = e^{-i\frac{qP}{\hbar}} e^{i\frac{pQ}{\hbar}}$$

$$U[p, q]^+ P U[p, q] = P + p$$

**WHY**

$$U[p, q]^+ Q U[p, q] = Q + q$$

$$|p, q\rangle = U[p, q]|\eta\rangle$$

$$\langle p, q | P | p, q \rangle = p + \langle \eta | P | \eta \rangle$$

$$\langle p, q | Q | p, q \rangle = q + \langle \eta | Q | \eta \rangle$$

Choose  $|\eta\rangle$  to be "physically centered":

$$\langle \eta | P | \eta \rangle = 0 = \langle \eta | Q | \eta \rangle \quad \therefore$$

$\langle p, q | P | p, q \rangle = p, \quad \langle p, q | Q | p, q \rangle = q$

$\{\frac{p}{q}\}$  is the mean value of  $\{\frac{P}{Q}\}$

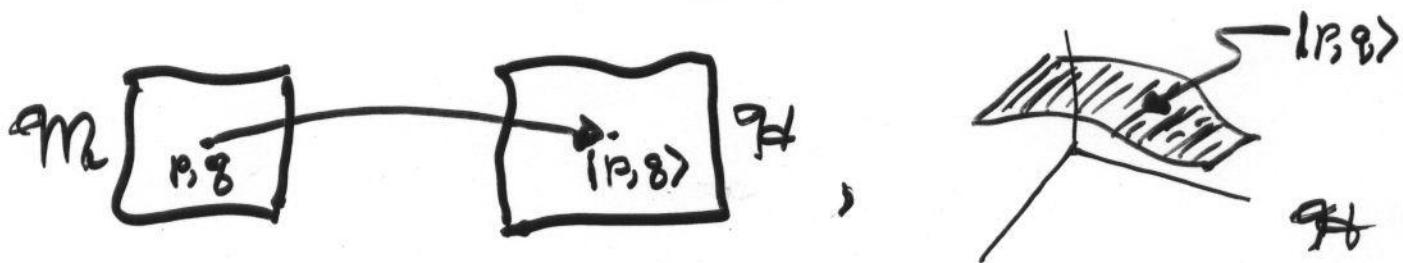
$$H(p, q) \equiv \langle p, q | H(P, Q) | p, q \rangle = \langle \eta | H(P + p, Q + q) | \eta \rangle$$

$$= H(p, q) + O(\hbar; p, q)$$

provided  $\langle \eta | (P^2 + Q^2) | \eta \rangle = O(\hbar)$

$H(p, q)$  is an " $\hbar$ -augmented"  
classical Hamiltonian

Interpret  $(p, q)$  as variables in classical Phase space  $\mathcal{M}$



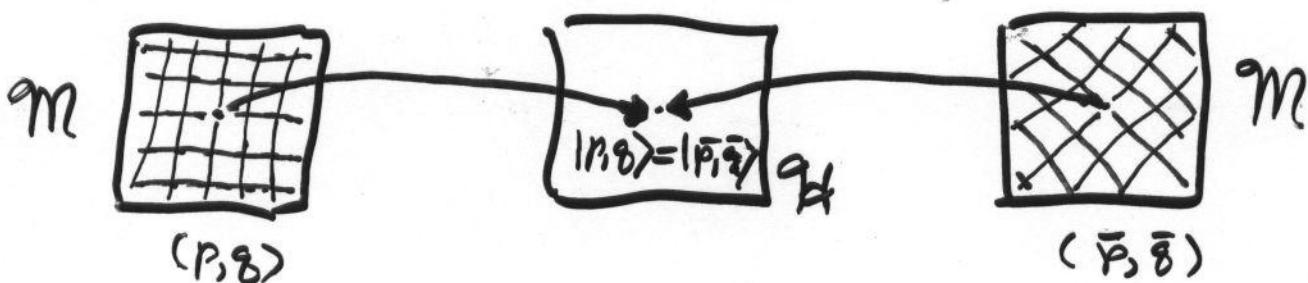
New canonical coordinates

$$\bar{p} = \bar{p}(p, q), \quad \bar{q} = \bar{q}(p, q)$$

$$\begin{aligned} pdg &= \bar{p} d\bar{q} + d\bar{F}(\bar{q}, \bar{p}) \\ pdg &= \bar{p} d\bar{q} + d\bar{G}(\bar{p}, \bar{q}) \end{aligned} \quad \left. \begin{array}{l} d\bar{p} d\bar{q} \\ = d\bar{p} d\bar{q} \end{array} \right.$$

$$p = p(\bar{p}, \bar{q}), \quad q = q(\bar{p}, \bar{q})$$

$$|p, q\rangle = |\bar{p}(\bar{p}, \bar{q}), \bar{q}(\bar{p}, \bar{q})\rangle \equiv |\bar{p}, \bar{q}\rangle$$



Geometry of coherent states

- Symplectic geometry of  $\{|p, q\rangle\}$

$\textcircled{1} \equiv i\hbar \langle p, q | d | p, q \rangle \quad (\text{one form})$

$$i\hbar d(e^{-i\bar{q}P/\hbar} e^{i\bar{p}Q/\hbar}) |\eta\rangle$$

$$= [e^{-i\bar{q}P/\hbar} (Pdg - Qdp) e^{i\bar{p}Q/\hbar}] |\eta\rangle$$

$$\Theta = \langle \eta | e^{-ipQ/\hbar} (Pdg - Qdp) e^{ipQ/\hbar} | \eta \rangle$$

$$= \langle \eta | [(P+p)dg - Qdp] | \eta \rangle$$

$$= pdg + \langle \eta | P | \eta \rangle dg - \langle \eta | Q | \eta \rangle dp$$

choose  $|\eta\rangle$  as physically centered

$$\boxed{\Theta(p, q) = i\hbar \langle p, q | d | p, q \rangle = pdg}$$

- Riemannian geometry of  $\{|p, q\rangle\}$

Background

Let  $|+\rangle$  and  $|-\rangle$  be unit vectors

$$d^2 \equiv \| |+\rangle - |-\rangle \| ^2 = 2[1 - \text{Re} \langle +|-\rangle]$$

$$d_{\text{RAY}}^2 \equiv \min_{\alpha} \| |+\rangle - e^{i\alpha} |-\rangle \| ^2 = 2[1 - |\langle +|-\rangle|]$$

$$= 2 \left[ 1 - \left\{ \left[ 1 - \frac{1}{2} \| |+\rangle - |-\rangle \| ^2 \right]^2 + \frac{1}{4} |\langle +|-\rangle - \langle -|+\rangle|^2 \right\}^{\frac{1}{2}} \right]$$

Specialize to  $|+\rangle = |-\rangle + d|-\rangle$

$$\text{N.B. } \langle -|d|-\rangle^* = -\langle -|d|-\rangle$$

$$d_{\text{RAY}}^2 = \| d|-\rangle \| ^2 - |\langle -|d|-\rangle|^2$$

Choose  $|-\rangle = |p, q\rangle$  (multiply by  $2\hbar$ )

$$\boxed{d\sigma^2(p, q) \equiv 2\hbar \left[ \| d|Rg\rangle \| ^2 - |\langle Rg|d|Rg\rangle|^2 \right]}$$

Notation:  $\langle A \rangle \equiv \langle n | A | n \rangle$

Canonical coherent states:

$$|p, q\rangle = e^{-i\theta P/\hbar} e^{ipQ/\hbar} |n\rangle$$

$$d\sigma^2 = \frac{2}{\hbar} [\langle Q^2 \rangle dp^2 + \langle PQ + QP \rangle dp dq + \langle P^2 \rangle dq^2]$$

Two-dimensional flat space for any  $|n\rangle$

Harmonic oscillator ground state

$$|n\rangle = |0\rangle, (Q + iP)|0\rangle = 0$$

$$d\sigma^2(p, q) = dp^2 + dq^2$$

Cartesian coordinates

Geometry in different coordinates:

$$\mathbb{H}(\bar{p}, \bar{q}) = \bar{p} d\bar{q} + d\bar{E}(\bar{p}, \bar{q})$$

$$d\sigma^2(\bar{p}, \bar{q}) = A(\bar{p}, \bar{q}) d\bar{p}^2 + B(\bar{p}, \bar{q}) d\bar{p} d\bar{q} + C(\bar{p}, \bar{q}) d\bar{q}^2$$

still a flat phase space

Quantum mechanics gives a metric to classical phase space!

For canonical variables (i.e., kinematics)

P and Q, the result is flat space

What about other quantum variables?

# Spin coherent states

$$|\theta, \varphi\rangle = e^{-i\varphi S_3} e^{-i\theta S_2} |s\rangle$$

$$\Theta(\theta, \varphi) = i\hbar \langle \theta, \varphi | d | \theta, \varphi \rangle = s\hbar \cos \theta d\varphi$$

canonical variables

$$p = \sqrt{s\hbar} \cos \theta, \quad q = \sqrt{s\hbar} \varphi$$

$$\Theta(p, q) = pdq$$

$$d\sigma^2(\theta, \varphi) = s\hbar [d\theta^2 + \sin^2 \theta d\varphi^2]$$

$$d\sigma^2(p, q) = \frac{dp^2}{(1 - p^2/s\hbar)} + (1 - p^2/s\hbar) dq^2$$

sphere of radius  $\sqrt{s\hbar}$ ,  $s = \frac{1}{2}, 1, \frac{3}{2}, \dots$   
(constant positive curvature)

# Affine coherent states

$$|p, q\rangle = e^{ipq/\hbar} e^{-i\ln q D/\hbar} |n\rangle, \quad q > 0$$

$$\Theta(p, q) = i\hbar \langle p, q | d | p, q \rangle = pdq$$

$$d\sigma^2(p, q) = (\beta\hbar)^{-1} q^2 dp^2 + (\beta\hbar) q^{-2} dq^2$$

pseudo sphere (Poincaré plane)

(constant negative curvature)

Exercise:

Find  $\Theta$  and  $d\sigma^2$  for non-group  $\{|z\rangle\}$

(leads to a non-constant curvature!)

# Action principles of physics

## - Classical action (phase space)

$$1) \quad I_c = \int_0^T [p\dot{q} - H(p, q)] dt$$

$\delta I_c = 0$  holding  $q(0)$  and  $q(T)$  fixed

$$\delta I_c = \int \left[ (\dot{q} - \frac{\partial H}{\partial p}) \delta p - (\dot{p} + \frac{\partial H}{\partial q}) \delta q \right] dt$$

$$+ p \delta q \Big|_0^T$$

$$2) \quad I_c = \int_0^T [-q\dot{p} - H(p, q)] dt$$

$$\delta I_c = \int \left[ (\dot{q} - \frac{\partial H}{\partial p}) \delta p - (\dot{p} + \frac{\partial H}{\partial q}) \delta q \right] dt$$

$$- q \delta p \Big|_0^T$$

$$3) \quad I_c = \int_0^T \left[ \frac{1}{2}(p\dot{q} - q\dot{p}) - H(p, q) \right] dt$$

$$\delta I_c = \int \left[ (\dot{q} - \frac{\partial H}{\partial p}) \delta p - (\dot{p} + \frac{\partial H}{\partial q}) \delta q \right] dt$$

$$+ \frac{1}{2}(p \delta q - q \delta p) \Big|_0^T$$

$$4) \quad I_c = \int_0^T [p\dot{q} + q\dot{p} - H(p, q)] dt$$

What values are held fixed at  
the end points  $t=0$  and  $t=T$ ?

— Quantum action (Hilbert space)

$$I_Q = \int_0^T \left[ \langle \psi | \left\{ i\hbar \frac{\partial}{\partial t} - H \right\} | \psi \rangle \right] dt$$

$$I_Q = \int_0^T \left\{ \left\{ \psi^*(x,t) i\hbar \frac{\partial \psi(x,t)}{\partial t} - \psi^*(x,t) H \psi(x,t) \right\} dx dt \right\}$$

$$\begin{aligned} \delta I_Q = & \iint \left[ i\hbar \frac{\partial \psi}{\partial t} - H \psi \right] \delta \psi^* dx dt \\ & + \iint \left[ -i\hbar \frac{\partial \psi^*}{\partial t} - H \psi^* \right] \delta \psi dx dt \\ & + i\hbar \int \psi^* \delta \psi dx \Big|_0^T \end{aligned}$$

What variables are held fixed here?

Two very different action principles of physics  
— Maybe not!

Restricted variational principle

Restrict  $|\psi(t)\rangle$  to  $|p(t), g(t)\rangle$

$$\begin{aligned} I_Q|_{p,g} &= \int \left[ i\hbar \langle p, g | \frac{d}{dt} | p, g \rangle - \langle p, g | H | p, g \rangle \right] dt \\ &= \int [p \dot{g} - H(p, g)] dt = I_C \end{aligned}$$

Classical action IS restricted form of quantum action!

We have called  $H(p, q)$  "h-augmented"—  
does it make a difference?

Generally, correction terms are tiny  
and have only very small contributions

But not always!

### Example

Classical Hamiltonian

$$H_c = \frac{p^2}{2m} - \frac{e^2}{q}, \quad q > 0$$

(all solutions with  $E < 0$  diverge)

Quantum Hamiltonian

$$H = \frac{p^2}{2m} - \frac{e^2}{Q}, \quad Q > 0$$

h-augmented classical Hamiltonian  
(using affine coherent states)

$$\begin{aligned} H &= \langle p, q | \frac{p^2}{2m} - \frac{e^2}{Q} | p, q \rangle \\ &= \langle n | \frac{1}{2m} \left( \frac{P}{q} + p \right)^2 - \frac{e^2}{qQ} | n \rangle \\ &= \frac{p^2}{2m} + \frac{\langle n | P^2 | n \rangle}{2m q^2} - \frac{e^2}{q} \langle n | Q | n \rangle \\ &\approx \frac{p^2}{2m} - \frac{e^2}{q} + \frac{\hbar^2}{am e^2} \frac{1}{q} \frac{e^2}{q} \\ &= \frac{p^2}{2m} - \frac{e^2}{q} + \left( \frac{a_0}{2q} \right) \frac{e^2}{q}, \quad a_0 = \text{Bohr radius} \end{aligned}$$

All classical singularities removed!