

## 2 PATH INTEGRALS

Schrödinger equation

$$i\hbar \frac{\partial \psi(x,t)}{\partial t} = \left\{ -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V(x) \right\} \psi(x,t)$$

$\psi(x,t)$ ,  $t > 0$ , determined by  $\psi(x,0)$

$$\psi(x'',t) = \int K(x'',t; x',0) \psi(x',0) dx'$$

$K$  is called the propagator

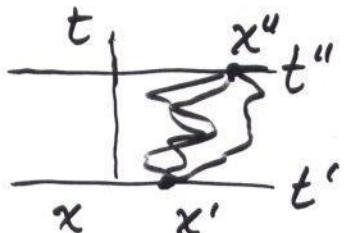
Solution of Schrödinger's equation  
with initial condition:  $\delta(x'' - x')$

Composition condition

$$K(x'',t''; x',t') \\ = \int K(x'',t''; x, t) K(x, t; x', t') dx$$

Feynman's 1st proposal (1948)

$$K(x'',t''; x',t') = \eta \int e^{i \int_{x'}^{x''} \left[ \frac{m}{2} \dot{x}(t)^2 - V(x(t)) \right] dt} dx$$



classical action

transcendental concept



Define by "lattice regularization"

Lattice regularization  $(t'' - t') = (N+1)\epsilon$  2-2

$$K(x'', t''; x', t') \\ = \lim_{N \rightarrow \infty} N \int e^{\frac{i}{\hbar} \sum_{k=0}^N \left\{ \frac{m(x_{k+1} - x_k)^2}{2\epsilon} - \epsilon V(x_k) \right\}} \prod_{k=1}^N dx_k$$

plausible idea, but needs a proof:

works if  $\frac{P^2}{2m} + V(Q)$  is essentially self adjoint on  $D(P^2) \cap D(V)$

works for  $V = Q^4$ , but fails for  $V = -Q^4$

Feynman-Kac formula (1950)

change  $t \rightarrow -it$ ,  $\hbar = 1$

$$\frac{\partial}{\partial t} P(x, t) = \left\{ \frac{1}{2m} \frac{\partial^2}{\partial x^2} - V(x) \right\} P(x, t)$$

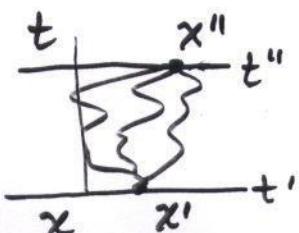
$$P(x'', t) = \int L(x'', t; x', 0) P(x', 0) dx'$$

$L(x'', t''; x', t')$

$$= q \int e^{- \int_{t'}^{t''} \left[ \frac{m}{2} \dot{x}(t)^2 + V(x(t)) \right] dt} dx$$

$$= \lim_{N \rightarrow \infty} N \int e^{- \sum_{k=0}^N \left[ \frac{m(x_{k+1} - x_k)^2}{2\epsilon} + \epsilon V(x_k) \right]} \prod_{k=1}^N dx_k$$

$$= \int e^{- \int_{t'}^{t''} V(x(t)) dt} d\mu_W(x)$$



$\mu_W$  genuine (pinned) measure concentrated on continuous but nowhere differentiable paths

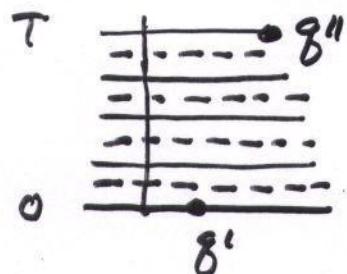
# Feynman's 2nd proposal (1950)<sup>2-3</sup>

$$i\hbar \frac{\partial \psi(x,t)}{\partial t} = \mathcal{H}\left(-i\hbar \frac{\partial}{\partial x}, x\right) \psi(x,t)$$

$$\psi(g'', T) = \int K(g'', T; g', 0) \psi(g'; 0) dg'$$

$$K(g'', T; g', 0) = \mathcal{W} \int e^{i\hbar \int_{g'}^{g''} [p(t) \dot{g}(t) - H(p(t), g(t))] dt} \mathcal{D}p \, dg$$

$g'' = g(T)$ ,  $g' = g(0)$ ; integrate over all  $p$



lattice formulation

$$T = (N+1)\epsilon$$

$$K(g'', T; g', 0)$$

$$= \lim_{N \rightarrow \infty} \int \int e^{i\hbar \sum_{k=0}^N [p_{k+\frac{1}{2}}(g_{k+1} - g_k) - \epsilon H(p_{k+\frac{1}{2}}, g_k)]} \times \prod_{k=0}^N \frac{dp_{k+\frac{1}{2}}}{2\pi\hbar} \prod_{k=0}^N dg_k$$

N.B. One more  $p$  integration than  
 $g$  integration. Required by

$$K(g'', t''; g', t') = \int K(g'', t''; g, t) K(g, t; g', t') dg$$

$$H(p_{k+\frac{1}{2}}, g_k) = \frac{\langle p_{k+\frac{1}{2}} | \mathcal{H} | g_k \rangle}{\langle p_{k+\frac{1}{2}} | g_k \rangle}$$

$$\text{If } \mathcal{H} = \frac{P^2}{2m} + V(Q)$$

then

$$H(p_{k+\frac{1}{2}}, g_k) = \frac{1}{2m} p_{k+\frac{1}{2}}^2 + V(g_k)$$

For such special Hamiltonians can integrate out  $p$  variables and recover original configuration space formulation

Proof of general expression ( $\hbar = 1$ )

$$K(q^*, T; q^*, 0) = \langle q^* | e^{-iT\mathcal{H}} | q^* \rangle$$

$$= \langle q^* | e^{-i\varepsilon\mathcal{H}} e^{-i\varepsilon\mathcal{H}} \dots e^{-i\varepsilon\mathcal{H}} | q^* \rangle$$

$$= \int \int \prod_{\ell=0}^N \langle q_{\ell+1} | e^{-i\varepsilon\mathcal{H}} | q_\ell \rangle \prod_{\ell=1}^N dq_\ell$$

$$= \int \int \prod_{\ell=0}^N \langle q_{\ell+1} | p_{\ell+\frac{1}{2}} \rangle \langle p_{\ell+\frac{1}{2}} | e^{-i\varepsilon\mathcal{H}} | q_\ell \rangle \prod_{\ell=0}^N dp_{\ell+\frac{1}{2}} \prod_{\ell=1}^N dq_\ell$$

$$\left( \begin{aligned} \langle p | e^{-i\varepsilon\mathcal{H}} | q \rangle &\simeq \langle p | [1 - i\varepsilon\mathcal{H}] | q \rangle \\ &= \langle p | q \rangle - i\varepsilon \langle p | \mathcal{H} | q \rangle \simeq \langle p | q \rangle e^{-i\varepsilon \frac{\langle p | \mathcal{H} | q \rangle}{\langle p | q \rangle}} \end{aligned} \right)$$

$$= \lim_{N \rightarrow \infty} \int \int \prod_{\ell=0}^N \langle q_{\ell+1} | p_{\ell+\frac{1}{2}} \times p_{\ell+\frac{1}{2}} | q_\ell \rangle e^{-i\varepsilon \langle p_{\ell+\frac{1}{2}} | \mathcal{H} | q_\ell \rangle / \langle p_{\ell+\frac{1}{2}} | q_\ell \rangle} \times \prod_{\ell=0}^N dp_{\ell+\frac{1}{2}} \prod_{\ell=1}^N dq_\ell$$

$$= \lim_{N \rightarrow \infty} \int \int e^{i\frac{\varepsilon}{\hbar} \sum_{\ell=0}^N [p_{\ell+\frac{1}{2}} (q_{\ell+1} - q_\ell) - \varepsilon H(p_{\ell+\frac{1}{2}}, q_\ell)]} \prod_{\ell=0}^N \frac{dp_{\ell+\frac{1}{2}}}{2\pi\hbar} \prod_{\ell=1}^N dq_\ell$$

Comments:

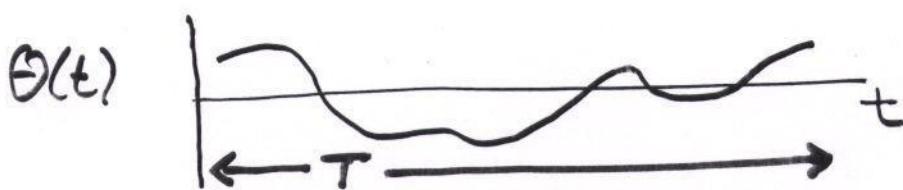
Meaning of  $p + q$  as sharp eigenvalues

Alternate "knowing"  $q, p, q, p, q, \dots$

Formulation implicitly states we know  $p + q$  alternately

Unnatural (like "poicules")

## Epicycles model planetary motion



$$\Theta(t) = \sum_n a_n \sin(\varphi_n + 2\pi n t/T)$$

make a model to "explain" behavior

Can we do "better"?

## Coherent state path integral (1960)

$$\Psi(p'', g'', T) = \langle p'', g'' | e^{-iHT/\hbar} | \psi \rangle$$

$$= \int \langle p'', g'' | e^{-iHT/\hbar} | p', g' \rangle \langle p', g' | \psi \rangle d\mu(p', g')$$

$$\Psi(p'', g'', T) = \int K(p'', g'', T; p', g'; 0) \Psi(p', g'; 0) dp' dg' / 2\pi\hbar$$

$$\langle p'', g'' | e^{-iHT} | p', g' \rangle = \langle p'', g'' | e^{-i\varepsilon H} e^{-i\varepsilon H} \dots e^{-i\varepsilon H} | p', g' \rangle$$

$$( \hbar = 1 , \quad T = (N+1)\varepsilon )$$

$$= \int \prod_{\ell=0}^N \langle p_{\ell+1}, g_{\ell+1} | e^{-i\varepsilon H} | p_\ell, g_\ell \rangle \prod_{\ell=1}^N d\mu(p_\ell, g_\ell)$$

$$\langle p, g | e^{-i\varepsilon H} | n, s \rangle \approx \langle p, g | [1 - i\varepsilon H] | n, s \rangle$$

$$= \langle p, g | n, s \rangle - i\varepsilon \langle p, g | H | n, s \rangle$$

$$\approx \langle p, g | n, s \rangle e^{-i\varepsilon H(p, g; n, s)}$$

$$\text{where } H(p, g; n, s) = \frac{\langle p, g | H | n, s \rangle}{\langle p, g | n, s \rangle}$$

$$K(p'', g'', T; p', g', 0)$$

$$= \lim_{N \rightarrow \infty} \int \int \prod_{\ell=0}^N \langle p_{\ell+1}, g_{\ell+1} | p_\ell, g_\ell \rangle e^{-i\varepsilon H(p_{\ell+1}, g_{\ell+1}; p_\ell, g_\ell)} \prod_{\ell=0}^N d\mu(p_\ell, g_\ell)$$

Interchange limit and  $\int \int$  (formally!) and write integrand for continuous and differentiable paths (restore  $\hbar$ )  
 $\langle 2|1 \rangle = 1 - \langle 2|(12) - 11 \rangle \approx e^{-i\varepsilon \langle 2|(12) - 11 \rangle}$   
 Use  $\langle 2|1 \rangle = 1 - \langle 2|(12) - 11 \rangle \approx e^{-i\varepsilon \langle 2|(12) - 11 \rangle}$

$$\begin{aligned} K(p'', g'', T; p', g', 0) &= \mathcal{W} \int e^{i\frac{\varepsilon}{\hbar} \int [i\hbar \langle p, g | d(p, g) - \langle p, g | H | p, g \rangle dt]} \\ &= \mathcal{W} \int e^{i\frac{\varepsilon}{\hbar} \int [p \dot{g} - H(p, g)] dt} \frac{dp dg}{\partial p \partial g} \end{aligned}$$

### Comments:

Meaning of  $p + g$  as mean values  
 Simultaneous "knowing" of  $p + g$  is O.K.  
Natural formulation

### Another formulation ( $\hbar = 1$ )

$$I = \int |p, g \times p, g| d\mu(p, g)$$

$$H = \int h(p, g) |p, g \times p, g| d\mu(p, g)$$

$$I - i\varepsilon H = \int [1 - i\varepsilon h(p, g)] |p, g \times p, g| d\mu(p, g)$$

$$e^{-i\varepsilon H} \approx \int e^{-i\varepsilon h(p, g)} |p, g \times p, g| d\mu(p, g)$$

$$\langle p'', g'' | e^{-i\varepsilon H T} | p', g' \rangle = \langle p'', g'' | e^{-i\varepsilon H} e^{-i\varepsilon H} \dots e^{-i\varepsilon H} | p', g' \rangle$$

$$= \lim_{N \rightarrow \infty} \int \prod_{k=0}^N \langle p_{k+1}, q_{k+1} | p_k, q_k \rangle \prod_{k=1}^N e^{-i\epsilon h(p_k, q_k)} d\mu(p_k, q_k)$$

Again, interchange orders and write formal path integral for continuous and differentiable paths

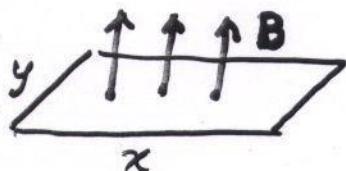
$$K(p'', q'', T; p', q', 0) = \mathcal{W} \int e^{i\hbar \int [i\dot{q} \langle p, q \rangle - h(p, q)] dt} \frac{\partial p}{\partial p} \frac{\partial q}{\partial q}$$

$$= \mathcal{W} \int e^{i\hbar \int [p \dot{q} - h(p, q)] dt} \frac{\partial p}{\partial p} \frac{\partial q}{\partial q}$$

Involves a different symbol (!) — added proof of formal nature of continuum expression

## Continuous-time regularization

Classical two-dimensional particle in a potential and a magnetic field



$$\underline{r} = (x, y, 0), \underline{B} = (0, 0, B)$$

$$\underline{A} = (0, Bx, 0), \nabla \times \underline{A} = \underline{B}$$

$$\mathbf{V} = V(x, y)$$

Newton's equation

$$m \ddot{\underline{r}} = -\nabla V + \dot{\underline{r}} \times \underline{B} \quad (\epsilon = c = 1)$$

Now let  $m \rightarrow 0$

$$\dot{\underline{r}} \times \underline{B} = \nabla V \iff B \dot{y} = \frac{\partial V}{\partial x}$$

$$B \dot{x} = -\frac{\partial V}{\partial y}$$

$$\sqrt{B}x \rightarrow p$$

$$\sqrt{B}y \rightarrow q$$

$$V(x, y) \rightarrow H(p, q)$$

$$\dot{q} = \frac{\partial H}{\partial p}$$

$$\dot{p} = -\frac{\partial H}{\partial q}$$

Hamilton's equations

Quantum propagator of two-dim. System

$$\psi(x'', y'', T) = \int K(x'', y'', T; x', y'; 0) \psi(x', y', 0) dx' dy'$$

$$K(x'', y'', T; x', y', 0)$$

$$= \eta \int e^{i\hbar \frac{1}{k} \int [A \cdot \dot{r} + \frac{m}{2} \dot{r}^2 - V(r)] dt} dx' dy$$

Limit  $m \rightarrow 0$ , or  $m \rightarrow i m \rightarrow 0$

$$= \lim_{m \rightarrow 0} \eta \int e^{i\hbar \int [A \cdot \dot{r} - V(r)] dt} - \frac{m}{2\hbar} \int (\dot{x}^2 + \dot{y}^2) dt dx' dy$$

Change names:  $\sqrt{B}x \rightarrow p$ ,  $\sqrt{B}y \rightarrow q$ ,  $V \rightarrow H$ ,  $m = B/\nu$

$$K(p'', q'', T; p', q', 0)$$

$$= \lim_{\nu \rightarrow \infty} \eta \int e^{i\hbar \int [pq - H(p, q)] dt} - \frac{1}{2\nu\hbar} \int (p^2 + q^2) dt dp dq$$

Feynman-Kac-like expression

$$= \lim_{\nu \rightarrow \infty} 2\pi e^{2T/\nu} \int e^{i\hbar \int [pdq - H(p, q)dt]} d\mu_w(p, q)$$

Well-defined path integral!

$$K(p'', q'', T; p', q', 0) \equiv \langle p'', q'' | e^{-iHt/\hbar} | p', q' \rangle$$

$$|p, q\rangle \equiv e^{-iqP/\hbar} e^{ipQ/\hbar} |0\rangle$$

$$[Q, P] = i\hbar I, (Q + iP)|0\rangle = 0$$

$$H = \int H(p, q) |p, q\rangle \langle p, q| dp dq / 2\pi\hbar$$

Solves Sch. equation for wide class of Hamiltonians

Covariance under coordinate transformations

$$\begin{aligned}
 K(p'', \dot{q}'', T; p', \dot{q}', 0) &= \langle p'', \dot{q}'' | e^{-i\langle T \rangle H/\hbar} | p', \dot{q}' \rangle \\
 &= \lim_{N \rightarrow \infty} \frac{1}{2\pi\hbar} e^{i\langle T \rangle / \hbar} \int e^{i\langle T \rangle / \hbar \int [p d\dot{q} - H(p, \dot{q}) dt]} d\mu_w^v(p, \dot{q}) \\
 \bar{K}(\bar{p}'', \bar{\dot{q}}'', T; \bar{p}', \bar{\dot{q}}', 0) &= \langle \bar{p}'', \bar{\dot{q}}'' | e^{-i\langle T \rangle H/\hbar} | \bar{p}', \bar{\dot{q}}' \rangle \\
 &= \lim_{N \rightarrow \infty} \frac{1}{2\pi\hbar} e^{i\langle T \rangle / \hbar} \int e^{i\langle T \rangle / \hbar \int [\bar{p} d\bar{\dot{q}} + d\bar{G}(\bar{p}, \bar{\dot{q}}) - \bar{H}(\bar{p}, \bar{\dot{q}}) dt]} d\mu_w^v(\bar{p}, \bar{\dot{q}})
 \end{aligned}$$

(uses Stratonovich (mid-point) rule)

$$H = \int H(p, \dot{q}) |p, \dot{q}\rangle \langle p, \dot{q}| dp d\dot{q} / 2\pi\hbar$$

$$\bar{H} = \int \bar{H}(\bar{p}, \bar{\dot{q}}) | \bar{p}, \bar{\dot{q}} \rangle \langle \bar{p}, \bar{\dot{q}} | d\bar{p} d\bar{\dot{q}} / 2\pi\hbar$$

$$d\mu_w^v(p, \dot{q}) = \eta e^{-\frac{1}{2\hbar} \int [p^2 + \dot{q}^2] dt} dp d\dot{q}$$

$$d\mu_w^v(\bar{p}, \bar{\dot{q}}) = \eta e^{-\frac{1}{2\hbar} \int [A\bar{p}^2 + B\bar{p}\bar{\dot{q}} + C\bar{\dot{q}}^2] dt} d\bar{p} d\bar{\dot{q}}$$

Coordinate covariance is natural because the mathematics involves the simple transcription  $(x, y) \rightarrow (p, \dot{q})$  of a genuine physical problem.

Conditionally convergent integral

$$\int_{-\infty}^{\infty} e^{iy^2/2} dy = \lim_{\substack{L_2 \rightarrow \infty \\ L_1 \rightarrow -\infty}} \int_{-L_1}^{L_2} e^{iy^2/2} dy$$

$$= \lim_{N \rightarrow \infty} \int_{-\infty}^{\infty} e^{iy^2/2} e^{-y^2/2N} dy = \lim_{N \rightarrow \infty} \sqrt{\frac{2\pi}{1/N - i}} = \sqrt{2\pi i}$$

# Convergence factors for path integrals

$$m \int e^{i/t_0 [p\dot{q} - H(p, q)] dt} \frac{\partial p}{\partial q}$$

No mathematics !

No physics !

Regularize

$$\begin{aligned} & \lim_{N \rightarrow \infty} m \int e^{i/t_0 [p\dot{q} - H(p, q)] dt} e^{-\frac{1}{2N} \int (\dot{p}^2 + \dot{q}^2) dt} \frac{\partial p}{\partial q} \\ &= \lim_{N \rightarrow \infty} 2\pi i \int e^{i/t_0 [p\dot{q} - H(p, q)] dt} du_w(p, q) \\ &= \langle p'', q'' | e^{-iT\lambda/t_0} | p', q' \rangle, \text{ canonical } [Q, P] = i\hbar I \end{aligned}$$

Other geometries

$$\lim_{N \rightarrow \infty} m \int e^{i/t_0 [p\dot{q} - H(p, q)] dt} e^{-\frac{1}{2N} \int \left[ \frac{\dot{p}^2}{1-p^2} + (1-p^2/\lambda)^2 \dot{q}^2 \right] dt} \frac{\partial p}{\partial q}$$

leads to spin kinematics:  $[S_j, S_k] = i\varepsilon_{jkl} S_l$

$$\lim_{N \rightarrow \infty} m \int e^{i/t_0 [p\dot{q} - H(p, q)] dt} e^{-\frac{1}{2N} \int [\beta^{-1} \dot{q}^2 \dot{p}^2 + \beta \dot{q}^{-2} \dot{q}^2] dt} \frac{\partial p}{\partial q}$$

leads to affine kinematics:  $[Q, D] = i\hbar Q, Q > 0$

- Phase space metric gives both mathematical & physical meaning to formal path integral
- Leads to coherent state representation directly

Quantization IS Geometry, After All !

# Evaluation of Integrals

$$F(y) = \frac{1}{2\pi} \int_0^{2\pi} e^{iy \cos \theta} d\theta$$

1) Power series:  $F(y) = 1 - \frac{y^2}{2!2} + \frac{3y^4}{4!8} - \dots$

2) Differential equations:  $yF''(y) + F'(y) + yF(y) = 0$   
 $F(0) = 1, F'(0) = 0$   
 $\Rightarrow F(y) = J_0(y)$  (Bessel function)

Path integrals

$$\begin{aligned} K(x'', T; x') &= \mathcal{N} \int e^{i\hbar \int \left[ \frac{m}{2} \dot{x}(t)^2 - V(x(t)) \right] dt} dx \\ &= \langle x'' | e^{-iHt/\hbar} | x' \rangle \\ &= \sum_{n=0}^{\infty} \psi_n(x'') e^{-iE_n t/\hbar} \psi_n^*(x') \end{aligned}$$

Sometimes can be summed  $\Rightarrow$  Table of Path Integrals

## Semiclassical Approximation of Path Integrals

Three representations

1)  $\psi(x) = \langle x | \psi \rangle$

2)  $\tilde{\psi}(p) = \langle p | \psi \rangle = \int \langle p | x \rangle \langle x | \psi \rangle dx$   
 $= \frac{1}{\sqrt{2\pi\hbar}} \int e^{-ipx/\hbar} \psi(x) dx$

3)  $\Psi(p, g) = \langle p, g | \psi \rangle = \int \langle p, g | x \rangle \langle x | \psi \rangle dx$   
 $= \left(\frac{\Omega}{\pi\hbar}\right)^{1/4} \int e^{-\frac{\Omega(g-x)^2}{2\hbar}} e^{ip(g-x)/\hbar} \psi(x) dx$

N.B.  $(\Omega Q + iP) |\eta \rangle = 0$

## Additional connections

$$a) \left(\frac{\Omega\hbar}{\Sigma}\right)^{1/4} \frac{1}{2\pi\hbar} \int \Psi(p, g) dp = \Psi(g)$$

$$b1) \lim_{\Sigma \rightarrow \infty} \left(\frac{\Sigma}{4\pi\hbar}\right)^{1/4} \Psi(p, g) = \Psi(g)$$

$$b2) \lim_{\Sigma \rightarrow 0} \left(\frac{1}{4\pi\hbar\Sigma}\right)^{1/4} \Psi(p, g) = e^{ipg/\hbar} \tilde{\Psi}(p)$$

One basic identity

$$\begin{aligned} \langle x'' | e^{-iTH/\hbar} | x' \rangle &\equiv \int \langle x'' | p'' \rangle \langle p'' | e^{-iTH/\hbar} | x' \rangle dp'' \\ &\equiv \int \langle x'' | p'', g'' \rangle \langle p'', g'' | e^{-iTH/\hbar} | x' \rangle dp'' dg'' / 2\pi\hbar \end{aligned}$$

Three basic semiclassical ( $= \infty$ ) approximations

$$\begin{aligned} \left[ \langle x'' | e^{-iTH/\hbar} | x' \rangle \right]_{sc} &\approx \int \langle x'' | p'' \rangle \left[ \langle p'' | e^{-iTH/\hbar} | x' \rangle \right]_{sc} dp'' \\ &\approx \int \langle x'' | p'', g'' \rangle \left[ \langle p'', g'' | e^{-iTH/\hbar} | x' \rangle \right]_{sc} dp'' dg'' / 2\pi\hbar \end{aligned}$$

e.p. = extremal paths

$$\left[ \langle x'' | e^{-iTH/\hbar} | x' \rangle \right]_{sc} = \sum_{e.p.} \frac{1}{\sqrt{2\pi\hbar [i\tilde{x}(\tau)]}} e^{iS_1(x''; x')/\hbar}$$

$$\left[ \langle p'' | e^{-iTH/\hbar} | x' \rangle \right]_{sc} = \sum_{e.p.} \frac{1}{\sqrt{2\pi\hbar [\tilde{p}(\tau)]}} e^{iS_2(p''; x')/\hbar}$$

$$\left[ \langle p'', g'' | e^{-iTH/\hbar} | x' \rangle \right]_{sc} = \frac{\sqrt{\Omega/\pi\hbar}}{\sqrt{[\tilde{p}(\tau) + i\Omega\tilde{x}(\tau)]}} e^{iS_3(p'', g''; x')/\hbar}$$

## Multiple extremal paths

$$\dot{x} = \frac{\partial H}{\partial p} \equiv H_p(p, x)$$

$$\dot{p} = -\frac{\partial H}{\partial x} \equiv -H_x(p, x)$$



paths differentiated by initial / final slope

Linearized deviations about extremal paths

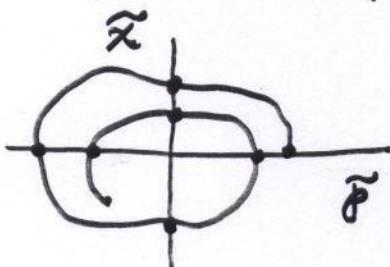
$$\dot{x} + \ddot{x} = H_p(p + \tilde{p}, x + \tilde{x}), \quad \dot{p} + \ddot{p} = -H_x(p + \tilde{p}, x + \tilde{x})$$

$$\ddot{x} = H_{pp}(p, x) \tilde{p} + H_{px}(p, x) \tilde{x}$$

$$\ddot{p} = -H_{xp}(p, x) \tilde{p} - H_{xx}(p, x) \tilde{x}$$

Solution

$$\begin{pmatrix} \tilde{x}(T) \\ \tilde{p}(T) \end{pmatrix} = \begin{pmatrix} A(T) & B(T) \\ C(T) & D(T) \end{pmatrix} \begin{pmatrix} \tilde{x}(0) = 0 \\ \tilde{p}(0) = 1 \end{pmatrix}$$



Meaning : multiple solutions

caustic :  $\tilde{x}(T) = 0$

pseudocaustic :  $\tilde{p}(T) = 0$

N.B. Can never have  $\tilde{x}(T) = 0$  and  $\tilde{p}(T) = 0$

Coherent state "se" approximation involves

$$\frac{1}{\sqrt{[\tilde{p}(T) + i\Omega \tilde{x}(T)]}}$$

and overcomes problem of multiple solutions

N.B. "dp" incorporates multiple extremal paths

Used in oil exploration & in  
under water detection