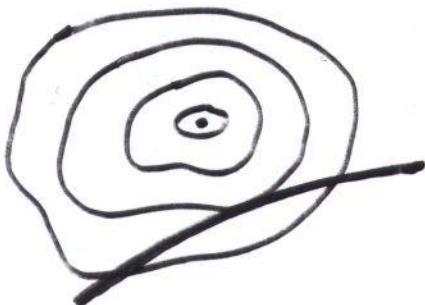


3. QUANTUM CONSTRAINTS

The Role of Lagrange Multipliers



height: $f(x, y)$

extremal: $\vec{\nabla}f(x, y) = 0$

constraint: $\varphi(x, y) = 0$

compatibility: $\vec{\nabla}f \propto \vec{\nabla}\varphi$

set of equations: $\vec{\nabla}f(x, y) = \lambda \vec{\nabla}\varphi(x, y), \varphi(x, y) = 0$

Classical Mechanics with Constraints

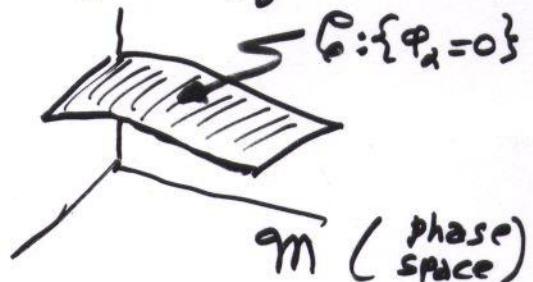
$$I = \int [p_j \dot{q}_j - H(p, q) - \lambda^\alpha \varphi_\alpha(p, q)] dt \quad \begin{matrix} 1 \leq j \leq J \\ 1 \leq \alpha \leq A \end{matrix}$$

e.o.m. $\ddot{q}_j = \frac{\partial H}{\partial p_j} + \lambda^\alpha \frac{\partial \varphi_\alpha}{\partial p_j}, \quad \dot{p}_j = -\frac{\partial H}{\partial q_j} - \lambda^\alpha \frac{\partial \varphi_\alpha}{\partial q_j}, \quad \varphi_\alpha = 0$

$$\dot{q}_j = \{q_j, H\} + \lambda^\alpha \{q_j, \varphi_\alpha\}$$

$$\dot{p}_j = \{p_j, H\} + \lambda^\alpha \{p_j, \varphi_\alpha\}$$

$$\varphi_\alpha = 0$$



Consistency requirement

$$\dot{\varphi}_\alpha(p, q) = \{\varphi_\alpha, H\} + \lambda^\beta \{\varphi_\alpha, \varphi_\beta\} = 0$$

There are two extreme cases:

First class start on G , stay on G freely

$$\{\varphi_\alpha, \varphi_\beta\} = C_{\alpha\beta}^\gamma \varphi_\gamma$$

$$\{\varphi_\alpha, H\} = h_\alpha^\beta \varphi_\beta$$

- a) $\mathcal{C}_{\alpha\beta}^\gamma$ numbers: closed 1st class (YM)
 b) $\mathcal{C}_{\alpha\beta}^\gamma$ functions: open 1st class (GR)

N.B. $\{\lambda^\alpha\}$ are undetermined by e.o.m.
 Must choose $\{\lambda^\alpha\}$ ("gauge") to solve e.o.m.

Second class start on \mathcal{G} , stay on \mathcal{G} by force

e.g., $\{\varphi_\alpha, \varphi_\beta\} \neq 0$ on \mathcal{G} , and has inverse on \mathcal{G}

$$\lambda^\beta = -\{\varphi_\alpha, \varphi_\beta\}^{-1}\{\varphi_\alpha, H\}$$

N.B. $\{\lambda^\alpha\}$ are determined by e.o.m.

- Also mixed 1st class & 2nd class systems

Quantization

1. Quantize before reduction
2. Reduce before quantization

1. Dirac

$$\varphi_\alpha(p, q) \rightarrow \bar{\Phi}_\alpha(p, q)$$

$$\bar{\Phi}_\alpha |\psi_{phys}\rangle = 0, \quad |\psi_{phys}\rangle \in \mathcal{H}_{phys} \subset \mathcal{H}$$

$$a) [\bar{\Phi}_\alpha, \bar{\Phi}_\beta] |\psi_{phys}\rangle = 0 ? , \quad b) \langle \psi_{phys} | \psi_{phys} \rangle < \infty ?$$

Works for some 1st class systems

2. Faddeev

$$m \int e^{i\frac{1}{\hbar} \int [p_j \dot{q}^j - H(p, q) - \lambda^\alpha \varphi_\alpha(p, q)] dt} \frac{\delta p_i \delta q_j \delta \lambda^\alpha}{\partial p_i \partial q_j \partial \lambda^\alpha}$$

$$= m' \int e^{i\frac{1}{\hbar} \int [p_j \dot{g}^j - H(p, g)] dt} \delta\{\varphi\} dp dg$$

Most likely diverges — choose a (dynamical) gauge



gauge equivalent sets of variables

$$\{x^\alpha, \varphi_\beta\} = \Delta^\alpha_\beta$$

$$\det(\Delta^\alpha_\beta) \neq 0$$

$$m' \int e^{i\frac{1}{\hbar} \int [p_j \dot{g}^j - H(p, g)] dt} \delta\{x\} \det\{x, \varphi\} \delta\{\varphi\} dp dg$$

$$= m' \int e^{i\frac{1}{\hbar} \int [p_B^* \dot{g}^* - H^*(p^*, g^*)] dt} \delta p^* dg^*$$

- Do quantization and reduction commute?

Consider the following example:

$$I = \int [p \dot{g} - \lambda(p^2 + g^4 - E)] dt$$

question: What E values yield a quantization?

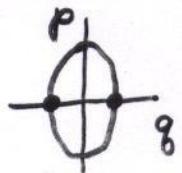
1. Dirac

$$(P^2 + Q^4 - E) |\Psi_{phys}\rangle = 0$$

Therefore, $E \in \{E_n\}$, set of eigenvalues, work

2. Faddeev

$$m \int e^{i\frac{1}{\hbar} \int p \dot{g} dt} \delta\{p^2 + g^4 - E\} dp dg$$



gauge: $x = -p = 0$, $\{x, \varphi\} = 4g^3$

$$I = m' \int e^{i\frac{1}{\hbar} \int p \dot{g} dt} \delta\{p\} (\Pi_t 4g^3) \delta\{p^2 + g^4 - E\} dp dg$$

$$= m' \int (\Pi_t 4g^3) \delta\{g^4 - E\} \Pi_t dg$$

i) $g > 0$: $I = \text{const.} > 0$; ii) $g \geq 0$: $I = 0$

Result: No restriction on E

before proceeding we introduce —

Another Property of Coherent States: Reproducing Kernel Hilbert Spaces

$\{|\ell\rangle\}$, set of continuous vectors that spans \mathcal{H}

Two elements in a dense set of vectors:

$$|\psi\rangle \equiv \sum_{j=1}^J \alpha_j |\ell_j\rangle , J < \infty$$

$$|\varphi\rangle \equiv \sum_{k=1}^K \beta_k |\ell_{(k)}\rangle , K < \infty$$

Functional representation:

$$\psi(\ell) \equiv \langle \ell | \psi \rangle = \sum_{j=1}^J \alpha_j \langle \ell | \ell_j \rangle$$

$$\varphi(\ell) \equiv \langle \ell | \varphi \rangle = \sum_{k=1}^K \beta_k \langle \ell | \ell_{(k)} \rangle$$

Inner product:

$$(\psi, \varphi) = \langle \psi | \varphi \rangle = \sum_{j,k=1}^{J,K} \alpha_j^* \beta_k \langle \ell_j | \ell_{(k)} \rangle$$

Complete space in norm $\|\psi\| = (\psi, \psi)^{1/2}$

N.B. All from Reproducing Kernel: $\langle \ell' | \ell \rangle$

Criteria for a kernel $\mathcal{K}(\ell'; \ell)$ to work:

1) continuity in ℓ' & ℓ

$$2) \sum_{j,k=1}^{J,K} \alpha_j^* \alpha_k \mathcal{K}(\ell_j; \ell_k) \geq 0$$

N.B. Let $\mathcal{K}_c(\ell'; \ell) = c \mathcal{K}(\ell'; \ell)$, $c > 0$

$$\psi_c(\ell) = \sum \alpha_j \mathcal{K}_c(\ell; \ell_j) = c \psi(\ell); \varphi_c(\ell) = c \varphi(\ell)$$

$$(\psi_c, \varphi_c)_c = \sum \alpha_j^* \beta_k \mathcal{K}_c(\ell_j; \ell_{(k)}) = c (\psi, \varphi)$$

Same space of functions; different inner prod.

Projection Operator Method

Quantize first: $\varphi_\alpha(P, q) \rightarrow \Phi_\alpha(P, Q)$

focus on $\sum_x \Phi_\alpha^2$ and note that

$$\Phi_\alpha |\psi_{\text{phys}}\rangle = 0 \iff \sum_x \Phi_\alpha^2 |\psi_{\text{phys}}\rangle = 0$$

Extend the Dirac Procedure as follows:

Projection operator: $E = E^2 = E^\dagger$; choose

$$E = E(\sum_x \Phi_\alpha^2 \leq \delta^2) \equiv \int_0^{\delta^2} dE (\sum_x \Phi_\alpha^2)$$

$$\psi_{\text{phys}} = E \psi \quad (\text{regularized by } \delta^2)$$

Final step: Reduce δ appropriately!

Three basic examples:

$$\textcircled{1} \quad E(J_1^2 + J_2^2 + J_3^2 \leq \hbar^2/2) \Rightarrow J_k = 0 \text{ all } k$$

$$\textcircled{2} \quad E(P^2 + Q^2 \leq \hbar) \Rightarrow (Q + iP) |\psi_{\text{phys}}\rangle = 0$$

if P & Q irreducible, $E = 10 \times 01$

Remark: Examples $\textcircled{1}$ & $\textcircled{2}$ have operators with discrete spectrum

$\textcircled{1}$ spectrum includes zero (1st class constraints)

$\textcircled{2}$ spectrum excludes zero (2nd class constraints)

$$\textcircled{3} \quad E(Q^2 \leq \delta^2), \text{ zero in continuous spectrum}$$

Sketch of issues:

$$\langle \varphi | E(Q^2 \leq \delta^2) | \varphi \rangle = \int_{-\delta}^{\delta} |\varphi(x)|^2 dx \xrightarrow{\delta \rightarrow 0} 0$$

$$\frac{1}{2\delta} \langle \varphi | E(Q^2 \leq \delta^2) | \varphi \rangle = \frac{1}{2\delta} \int_{-\delta}^{\delta} |\varphi(x)|^2 dx \xrightarrow{\delta \rightarrow 0} |\varphi(0)|^2$$

provided $\varphi(x)$ continuous at zero

A more careful way to proceed:
use coherent states to take a
form limit

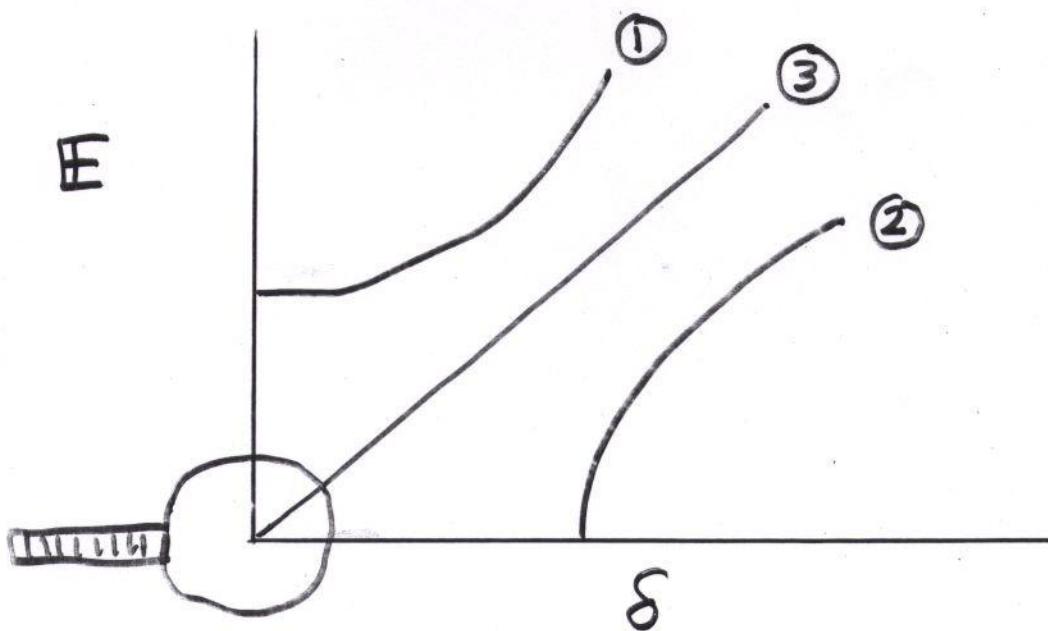
$$\begin{aligned} K(p', q'; p, q) &= \frac{\langle p', q' | E(Q^2 \leq \delta^2) | p, q \rangle}{\langle 0, 0 | E(Q^2 \leq \delta^2) | 0, 0 \rangle} \\ &= \frac{\int_{-\delta}^{\delta} \langle p', q' | x \rangle \langle x | p, q \rangle dx}{\int_{-\delta}^{\delta} \langle 0, 0 | x \rangle \langle x | 0, 0 \rangle dx} \end{aligned}$$

As $\delta \rightarrow 0$,

$$K_0(p', q'; p, q) \equiv \frac{\langle p', q' | x=0 \rangle \langle x=0 | p, q \rangle}{\langle 0, 0 | x=0 \rangle \langle x=0 | 0, 0 \rangle}$$

Needs only a continuous fiducial vector $\eta(x)$.
Here, \mathcal{H}_0 is a RK for a one-dimensional
physical Hilbert space, $\mathcal{H}_{\text{phys}}$

Rough graphical summary



An operator identity: Integral representation for \mathbb{E}

$$\mathbb{E}(\sum_{\alpha} \Phi_{\alpha}^2 \leq \delta^2) = \int \prod_{\alpha} e^{-i \int_{t_1}^{t_2} \lambda_{\alpha}(t) \Phi_{\alpha} dt} dR(\lambda)$$

$R(\lambda)$ depends on A , $t_2 > t_1$, and δ^2 — it is completely independent of the set $\{\Phi_{\alpha}\}_{\alpha=1}^A$

The integral representation can be used in path integral constructions such as

$$\langle p_i^u g^u | \mathbb{E} e^{-i (\mathcal{E} H \mathcal{E}) T / \hbar} \mathbb{E} | p_i' g' \rangle$$

$$= \mathcal{M} \int e^{i \frac{q_i}{\hbar} [p_i \dot{g}^u - H(p, q) - \lambda^u \Phi_u(p, q)] t} Dp Dq dR(\lambda)$$

valid for 1st & 2nd class constraints

Comparison with Traditional Methods

Traditional Methods

- Gauge fixing
- Faddeev-Popov det.
- Grinov ambiguity
- Auxiliary variables
e.g., ghosts
- Dirac brackets

:

Projection Operator Method

- { No gauge fixing
- { No auxiliary variables
- { No need to eliminate 2nd class constraints

:

Two advanced examples:

④ Classical

$$I = \int [P\dot{q} - \lambda q^3(2-q)] dt$$

$$\dot{q}=0, \dot{P}=-3\lambda q^2(2-q) + \lambda q^3, q^3(2-q)=0$$

Solution of e.o.m.

$$q(t) = q(0), q(0) = 0, 2 \text{ since } q^3(2-q) = 0$$

$$P(t) = P(0) - 3q^2(0)(2-q(0)) \int_0^t \lambda(t') dt' \\ + q^3(0) \int_0^t \lambda(t') dt'$$

$$\therefore q(t) = 0, P(t) = P(0) \text{ when } q(0) = 0$$

$$q(t) = 2, P(t) = P(0) + 8 \int_0^t \lambda(t') dt' \text{ if } q(0) = 2$$

The 'physical' reduced classical phase space is

$$(P, q) = (P, 0) + (P, 2)$$

N.B. P for $(P, 0)$ is gauge independent
 P for $(P, 2)$ is gauge dependent

N.B. The constraint at $q=2$ is called a regular constraint; the constraint at $q=0$ is called an irregular constraint

Quantum

$$\begin{aligned} E(Q^6(2-Q)^2 \leq \delta^2) &= E(-\delta \leq Q^3(2-Q) \leq \delta) \\ &= E(-\delta \leq Q^3(2-Q) \leq \delta), 0 < \delta \ll 1 \\ &= E(-\delta \leq 2Q^3 \leq \delta) + E(-\delta \leq 8(2-Q) \leq \delta) \\ &\equiv E_0(-\delta_0 \leq Q \leq \delta_0) + E_2(2-\delta_2 \leq Q \leq 2+\delta_2) \end{aligned}$$

where $\delta_0 \equiv (\delta/2)^{1/3}$, $\delta_2 \equiv \delta/8$

Use coherent state matrix elements to make Reproducing Kernels

$$\begin{aligned} \mathcal{K}(p'; g'; p, g) &\equiv \langle p'; g' | E_0 | p, g \rangle + \langle p'; g' | E_2 | p, g \rangle \\ &\equiv \mathcal{K}_0(p'; g'; p, g) + \mathcal{K}_2(p'; g'; p, g) \end{aligned}$$

The fact that $E_0 E_2 = 0$ leads to

$$\begin{aligned} \int \mathcal{K}_0(p'', g''; p, g) \mathcal{K}_2(p, g; p', g') d\mu(p, g) \\ = \langle p'', g'' | E_0 E_2 | p', g' \rangle = 0 \end{aligned}$$

Introduce the function ($a > 0, b > 0$)

$$A(p', g'; p, g) = \sqrt{a} \frac{\mathcal{K}_0(p', g'; p, g)}{\sqrt{\langle 0,0 | E_0 | 0,0 \rangle}} + \sqrt{b} \frac{\mathcal{K}_2(p', g'; p, g)}{\sqrt{\langle 0,2 | E_2 | 0,2 \rangle}}$$

and define the new Reproducing Kernel!

$$\begin{aligned} \hat{\mathcal{K}}(p'', g''; p', g') &\equiv \int A(p'', g''; p, g) \mathcal{K}(p, g; p', g') \\ &\quad \times d\mu(p, g) d\mu(p', g') \\ &= a \frac{\langle p'', g'' | E_0 | p', g' \rangle}{\langle 0,0 | E_0 | 0,0 \rangle} + b \frac{\langle p'', g'' | E_2 | p', g' \rangle}{\langle 0,2 | E_2 | 0,2 \rangle} \end{aligned}$$

Now, take limit $\delta \rightarrow 0$ to yield another RK

$$\tilde{\mathcal{K}}(p'', g''; p', g') = a \frac{\langle p'', g'' | x=0 \rangle \langle x=0 | p', g' \rangle}{\langle 0,0 | x=0 \rangle \langle x=0 | 0,0 \rangle} + b \frac{\langle p'', g'' | x=2 \rangle \langle x=2 | p', g' \rangle}{\langle 0,2 | x=2 \rangle \langle x=2 | 0,2 \rangle}$$

which defines a two-dimensional $\mathcal{H}_{\text{phys}}$

Specialize to

$$|P, g\rangle = e^{iPQ/\hbar} e^{-i\hbar P/\hbar} |0\rangle, \quad (Q+iP)|0\rangle = 0$$

which leads to

$$\tilde{\mathcal{K}}(P'', Q''; P', Q') = e^{-(Q''^2 + Q'^2)/2\hbar} \\ + e^{-i2(P'' - P')/\hbar} e^{-[(Q'' - 2)^2 + (Q' - 2)^2]/2\hbar}$$

N.B. In first term "Q=0", in second term "Q=2"

Observables in 'physical' Hilbert space

Observable Θ obeys $[\Theta, E] = 0$

Observable part of G , $G^E \equiv EGE$

Observable momentum & coordinate

$$P^E = EPE, \quad Q^E = EQE$$

Classical (" \hbar -augmented") momentum

$$P(P, g; \hbar) = \lim_{\delta \rightarrow 0} \frac{a\langle P, g | E_0 P E_0 | P, g \rangle + b\langle P, g | E_2 P E_2 | P, g \rangle}{a\langle P, g | E_0 | P, g \rangle + b\langle P, g | E_2 | P, g \rangle} = p$$

$$\therefore P^P(P, g) = \lim_{\hbar \rightarrow 0} P(P, g; \hbar) = p$$

meaning that the range of the variable $P^P(P, g)$ is the same as p , i.e., all of \mathbb{R}

Classical (" \hbar -augmented") coordinate

$$Q^P(P, g; \hbar) = \lim_{\delta \rightarrow 0} \frac{a\langle P, g | E_0 Q E_0 | P, g \rangle + b\langle P, g | E_2 Q E_2 | P, g \rangle}{a\langle P, g | E_0 | P, g \rangle + b\langle P, g | E_2 | P, g \rangle}$$

$$g^P(p, \theta; \hbar) = \frac{2be^{-(\theta-2)^2/\hbar}}{ae^{-\theta^2/\hbar} + be^{-(\theta-2)^2/\hbar}}$$

To obtain true classical limit

$$g^P(p, \theta) = \lim_{\hbar \rightarrow 0} g^P(p, \theta; \hbar) = \begin{cases} 0, & \theta < 1 \\ 2, & \theta > 1 \end{cases}$$

and we recover the original 'physical' reduced phase space :

$$(p, \theta) = (p, 0) \text{ or } (p, 2), p \in \mathbb{R}$$

(Observe that

$$g^P(p, 1) = \lim_{\hbar \rightarrow 0} g^P(p, 1; \hbar) = \frac{2b}{a+b}$$

holds only for a set of measure zero in the original classical phase space.)

⑤ An example with an "anomaly"

Classical: Three examples in parallel!

$$a) \varphi_k = j_k = \epsilon_{kem} p_e g_m = 0$$

$$\{j_k, j_\ell\} = \epsilon_{kem} j_m \quad \text{1st class (closed)}$$

$$b) \varphi_k = r_k \equiv f j_k = 0, f = (1-\alpha) + \alpha (p_k^2 + g_k^2)/\hbar$$

$$\{r_k, r_\ell\} = \epsilon_{kem} f(p, \theta) r_m \quad \text{1st class (open)}$$

$$c) \varphi_k = s_k \equiv g j_k = 0, g = (1 - 3\beta/2) + \beta \left[\frac{(p_1^2 + g_1^2)}{\hbar} + \frac{(p_2^2 + g_2^2)}{\hbar} \right]$$

$$\{s_k, s_\ell\} = \alpha_{kem}(p, \theta) s_m \quad \text{1st class (open)}$$

Quantum: Three examples in parallel

$$\vec{J}_k \rightarrow J_k = \epsilon_{kem} P_e Q_m = i\hbar a_e^+ \epsilon_{kem} a_m$$

$$N = a_e^+ a_e = N_1 + N_2 + N_3$$

$$F \rightarrow F = 1 + 2\alpha(1 + N) , [F, J_k] = 0$$

$$g \rightarrow G = 1 + \beta(2N_1 + N_2) , [G, J_k] \neq 0$$

$$N.B. [N, J_k] = [N, F] = [N, G] = 0$$

Restrict study to $N=2$ subspace

$$\left\{ |2,0,0\rangle, |0,2,0\rangle, |0,0,2\rangle \right\}$$

$$\left\{ |0,1,1\rangle, |1,1,0\rangle, |1,1,1,0\rangle \right\}$$

$$a) E(\sum J_k^2 \leq \hbar^2) = |0_J\rangle \langle 0_J| , \sum J_k^2 |0_J\rangle = 0$$

$$|0_J\rangle \propto |2,0,0\rangle + |0,2,0\rangle + |0,0,2\rangle$$

$$b) E(\sum R_k^2 = F^2 \sum J_k^2 \leq \hbar^2) = |0_R\rangle \langle 0_R| , \sum R_k^2 |0_R\rangle = 0$$

$$|0_R\rangle = |0_J\rangle \propto |2,0,0\rangle + |0,2,0\rangle + |0,0,2\rangle$$

Both $\sum J_k^2$ & $\sum R_k^2$ are 1st class quantum constraints because spectrum has zero

$$c) S_k = (G J_k + J_k G)/2$$

$$E(\sum S_k^2 \leq \hbar^2) = |0_S\rangle \langle 0_S|$$

$$\sum S_k^2 |0_S\rangle = (\hbar^2 \beta/2 + \dots) |0_S\rangle$$

for $0 < \beta \ll 1 \therefore$ 2nd class

$$|0_S\rangle \propto (1 - 2\beta + \dots) |2,0,0\rangle + (1 - \beta + \dots) |0,2,0\rangle + |0,0,2\rangle$$