

4. QUANTUM GRAVITY

Electromagnetism, classical

Gravity is a gauge theory somewhat like E+M

$$I = \int [A_{\mu,\nu} F^{\mu\nu} + \frac{1}{4} F_{\mu,\nu} F^{\mu\nu}] d^4x$$

$$F_{\mu\nu} = 0, \quad A_{\nu\mu} - A_{\mu\nu} = F_{\mu\nu}$$

$$\text{let } E^i = F^{0i}, \quad B^k = \frac{1}{2} \epsilon^{ijk} F_{ij}$$

$$I = \int [-E^i \dot{A}_i - \frac{1}{2} (E^i E^i + B^i B^i) - A_0 E_i^i] d^3x dt$$

Quantum ($\hbar = 1$)

$$\text{operators: } [E^i(x), A_k(y)] = i \delta_k^i \delta(x-y)$$

$$\text{C-numbers: } \xi = \xi_T + \xi_L, \quad \nabla \cdot \xi_T = 0 = \nabla \times \xi_L$$

$$\alpha = \alpha_T + \alpha_L, \quad \nabla \cdot \alpha_T = 0 = \nabla \times \alpha_L$$

$$|\xi, \alpha\rangle = e^{i \int [\alpha \cdot \xi - \xi \cdot A] d^3x} |0\rangle$$

$$\langle \xi'', \alpha'' | \xi', \alpha' \rangle = e^{i \sum_{J=L}^T \int [\xi_J'' \cdot \alpha_J' - \alpha_J'' \cdot \xi_J'] d^3x}$$

$$e^{-\frac{1}{4} \sum_{T,L} \left\{ \left| \frac{\xi_J'' - \xi_J'}{\omega} \right|^2 + \omega |\tilde{\alpha}_J'' - \tilde{\alpha}_J'|^2 \right\} d^3k}$$

$$\text{Use } b = \nabla \times \alpha = \nabla \times \alpha_T = b_T, \quad \omega^2 \tilde{\alpha}_T^2 = \tilde{b}_T^2, \quad \omega = |b_T|$$

Introduce dynamics plus constraint

$$\lim_{S \rightarrow 0} \frac{\langle \xi'', \alpha'' | e^{-iHt} E | \xi', \alpha' \rangle}{\langle 0, 0 | E | 0, 0 \rangle} = e^{i \sum_{J=T} \int [\xi_J'' \cdot \alpha_J'(t) - \alpha_J'' \cdot \xi_J'(t)] d^3x}$$

$$\times e^{-\frac{1}{4} \sum_{(x-y)^2} \left\{ \frac{[\xi_T''(x) - \xi_T'(x,t)] \cdot [\xi_T''(y) - \xi_T'(y,t)] + [\tilde{b}_T''(x) - \tilde{b}_T'(x,t)] \cdot [\tilde{b}_T''(y) - \tilde{b}_T'(y,t)]}{\omega^2} \right\}}$$

$$\times e^{-\frac{1}{2} \sum_{(x-y)^2} \left\{ \frac{\xi_L''(x) \cdot \xi_L''(y)}{\omega^2} + \frac{\xi_L'(x) \cdot \xi_L'(y)}{\omega^2} \right\} d^3x d^3y}$$

N.B. $\xi'_T(t)$, $\alpha'_T(t)$, $b'_T(t)$ evolve with classical dynamics

Classical gravity

$$\mathcal{I} = \frac{c^4}{16\pi G} \int \sqrt{-g} R(g) d^4x \quad , \quad \frac{c^4}{16\pi G} = 1 = c = \hbar$$

First order form (Palatini)

$$\mathcal{I} = \int \sqrt{-g} g^{\mu\nu} [\Gamma_{\mu\nu,\alpha}^\alpha - \Gamma_{\mu\alpha,\nu}^\alpha + \Gamma_{\mu\nu}^\alpha \Gamma_{\alpha\rho}^\beta - \Gamma_{\mu\rho}^\alpha \Gamma_{\nu\alpha}^\beta] d^4x$$

Vary $g^{\mu\nu}$ get $R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0$

Vary $\Gamma_{\mu\nu}^\alpha$ recover definition of $\Gamma_{\mu\nu}^\alpha(g)$

Make 3+1 split of "space" plus "time" (ADM)

$$\mu, \nu = 0, 1, 2, 3 ; i, j, k = 1, 2, 3$$

$${}^4g_{\mu\nu} = \left[\begin{array}{c|c} g_{ij} & N_i \\ \hline N_j & -(N^2 - N_i N^j) \end{array} \right], \quad g^{jk} g_{kj} = \delta^j_k \\ \{g_{ij}\} > 0$$

$$\pi^{ij} = \sqrt{-g} ({}^4\Gamma_{pq}^0 - g_{pq} {}^4\Gamma_{rs}^0 g^{rs}) g^{ip} g^{jq}$$

$$\boxed{\mathcal{I} = \int \sqrt{-4g} {}^4R d^4x \doteq \int [-g_{ab} \dot{\pi}^{ab} - N^a H_a - NH] d^3x dt}$$

$$\dot{g}_{ab} = \dots, \quad \dot{\pi}^{ab} = \dots,$$

$$H_a = -2\pi_a^b l_b = 0$$

$$H = \frac{1}{\sqrt{g}} [\pi_a^a \pi_a^b - \frac{1}{2} \pi_a^a \pi_b^b] + \sqrt{g} R = 0$$

Fundamental Poisson brackets

$$\{g_{ab}(x), g_{cd}(y)\} = 0 = \{\pi^{ab}(x), \pi^{cd}(y)\}$$

$$\{g_{ab}(x), \pi^{cd}(y)\} = \frac{1}{2} (\delta_a^c \delta_b^d + \delta_a^d \delta_b^c) \delta(x, y)$$

Constraint brackets

$$\{H_a(x), H_b(y)\} = \delta_{a,b}(x,y) H_b(x) - \delta_{b,a}(x,y) H_a(x)$$

$$\{H_a(x), H(y)\} = \delta_{a,b}(x,y) H(x)$$

$$\{H(x), H(y)\} = \delta_{a,b}(x,y) g^{ab}(x) H_b(x)$$

Leads to open 1st class constraint system

16 = (6+6+4) equations of motion for 16 variables (g_{ab}, π^{ab}, N^a, N)

However N^a, N are not determined by E.O.M.

They need to be chosen to find solution

Fixing N^a, N amounts to choosing future coordinate system

One possibility is to reduce to physical variables so solve constraints for 4 variables

(among g_{ab}, π^{ab}), then impose "coordinate conditions" (i.e., dynamical gauge fixing — the $X^a = 0$ of Faddeev), to eliminate 4 more leaving $6+6-4-4=4 = "2p"+"2g"$ variables. A difficult nonlinear problem; besides the remaining 2p, 2g are unlikely to be Cartesian. A major "road block"!

Kinematics

Projection operator method insists on no reduction before quantization. One must quantize all $g_{ij} \rightarrow \hat{g}_{ij}$, $\pi^{ij} \rightarrow \hat{\pi}^{ij}$, subject to $\{\hat{g}_{ij}\} > 0$, and then reduce.

Adopt standard commutation relations:

$$[\hat{g}_{ab}(x), \hat{\pi}^{cd}(y)] = \frac{i}{2} (\delta_a^c \delta_b^d + \delta_a^d \delta_b^c) \delta(x, y)$$

infinitely many unitarily inequivalent, irreducible representations

However, all representations imply that

$$e^{i \int u_{ab} \hat{\pi}^{ab} dx} \hat{g}_{cd}(x) e^{-i \int u_{ab} \hat{\pi}^{ab} dx}$$

$$= \hat{g}_{cd}(x) + u_{cd}(x) \equiv \hat{g}_{cd}^u : \text{no longer pos. def.}$$

HELP from a single degree of freedom

Recall affine variables

$$[Q, P] = i \cdot, \quad Q > 0$$

$$e^{iuP} Q e^{-iuP} = Q + u : \text{no longer pos. def.}$$

$$\text{solution } [Q, P]Q = [Q, D] = iQ, \quad Q > 0$$

$$D = \frac{1}{2}(QP + PQ), \quad e^{iuD} Q e^{-iuD} = e^u Q > 0$$



Suggests replacing $\hat{\pi}^{ab}(x)$ by

$$\pi_c^a(x) \equiv \hat{\pi}^{ab}(x) g_{bc}(x)$$

i.e., "lower" one index on momentum

$\pi_c^a(x)$ is called momentric (momentum + metric)

Leads to new classical Poisson brackets:

$$\{\pi_b^a(x), \pi_d^c(y)\} = \frac{1}{2} [\delta_b^c \pi_d^a(x) - \delta_d^a \pi_b^c(x)] \delta(x, y)$$

$$\{g_{ab}(x), \pi_d^c(y)\} = \frac{1}{2} [\delta_a^c g_{bd}(x) + \delta_b^c g_{ad}(x)] \delta(x, y)$$

$$\{g_{ab}(x), g_{cd}(y)\} = 0, \quad \{g_{ab}(x)\} > 0$$

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New classical variables are as good as original variables because

$$\pi^{ab}(x) = g^{ac}(x) \pi^a_c(x)$$

New Poisson brackets close to 'Lie' algebra

Let us quantize new variables: g_{ab}, π^a_c

Quantum commutators

$$[\hat{\pi}^a_b(x), \hat{\pi}^c_d(y)] = \frac{i}{2} [\delta^c_b \hat{\pi}^a_d(x) - \delta^a_d \hat{\pi}^c_b(x)] \delta(x,y)$$

$$[\hat{g}_{ab}(x), \hat{\pi}^c_d(y)] = \frac{i}{2} [\delta^c_a \hat{g}_{bd}(x) + \delta^c_b \hat{g}_{ad}(x)] \delta(x,y)$$

$$[\hat{g}_{ab}(x), \hat{g}_{cd}(y)] = 0 , \quad \{ \hat{g}_{ab}(x) \} > 0$$

Direct algebraic consequence

$$e^{i \int y^a_b \hat{\pi}^b_a d^3 y} \hat{g}_{cd}(x) e^{-i \int y^a_b \hat{\pi}^b_a d^3 y}$$

$$= \{ e^{y(x)/2} \}_c^e \hat{g}_{ef}(x) \{ e^{y(x)/2} \}_d^f \quad \text{😊}$$

which maintains positive definite property

and leads to self-adjoint representations

$$\text{N.B. } \hat{\pi}^{ab}(x) = \left[\frac{1}{2} [g^{bc}(x) \hat{\pi}^a_c(x) + \hat{\pi}^a_c(x) g^{bc}(x)] \right]$$

is not an operator (even after smearing)

Affine group algebra derives from

$$U[\pi, y] \equiv e^{i \int \pi^{ab}(y) \hat{g}_{ab}(y) d^3 y} e^{-i \int y^a_b(y) \hat{\pi}^b_a(y) d^3 y}$$

and linear combinations of smooth functions

$\pi^{ab} + y^a_b$ of, e.g., compact support.

Choose a representation

There exist infinitely many unitarily inequivalent, irreducible representations Equivalent to choosing $|\eta\rangle$ and defining

$$|\pi, \gamma\rangle \equiv e^{i\int \pi^{ab}(y) \hat{g}_{ab}(y) dy} e^{-i\int \gamma^a_b(y) \hat{\pi}_a^b(y) dy} |\eta\rangle$$

Basic properties of $|\eta\rangle$

$\langle \eta | \hat{g}_{ab}(x) | \eta \rangle \equiv \tilde{g}_{ab}(x)$ fixes topology and asymptotic nature of spacelike surface.

$$\langle \eta | \hat{\pi}_b^a(x) | \eta \rangle = 0$$

Algebraic consequences

$$\langle \pi, \gamma | \hat{g}_{ab}(x) | \pi, \gamma \rangle = \{e^{\beta(x)\gamma_c}\}_a^c \tilde{g}_{cd}(x) \{e^{\beta(x)}\}_b^d \equiv g_{ab}(x)$$

$$\langle \pi, \gamma | \hat{\pi}_c^a(x) | \pi, \gamma \rangle = \pi^{ab}(x) g_{bc}(x) \equiv \pi_c^a(x)$$

HELP from a single degree of freedom

$$|p, g\rangle = e^{ipQ} e^{-i\ln(g)D} |\eta\rangle, \quad Q > 0, \quad g > 0$$

$$\langle \eta | Q | \eta \rangle = 1, \quad \langle \eta | D | \eta \rangle = 0$$

$$\langle p, g | Q | p, g \rangle = g, \quad \langle p, g | D | p, g \rangle = pg$$

$$\langle p'', g'' | p', g' \rangle = \left\{ \frac{g''^{-1/2} g'^{1/2}}{\left[\frac{1}{2}(g''^{-1} + g'^{-1}) + \frac{i}{2\beta}(p'' - p) \right]} \right\}^{2P}$$

Save for numerator, expression is an analytic function of $g'' + (i/\beta)p''$

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Full coherent state overlap function

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$$\langle \pi''\gamma'' | \pi'\gamma' \rangle =$$

$$\exp \left[-2 \int b(x) d^3x \ln \left(\frac{\det \left\{ \frac{1}{2} [g''^{ab}_{xx} + g'^{ab}(x)] + \frac{i}{2b(x)} [\pi''^{ab}_{xx} - \pi'^{ab}(x)] \right\}}{\det \{g''^{ab}\}^{1/2} \det \{g'^{ab}(x)\}^{1/2}} \right) \right]$$

$$\equiv \langle \pi''\gamma'' | \pi',\gamma' \rangle$$

- 1) • Depends on $0 < b(x) < \infty$, a scalar density with dimensions L^{-3}
- Different $b(x)$ correspond to inequivalent field operator representations
- Expect $b(x)$ will disappear when constraints are fully enforced and final operator representation is attained

2) Only depends on $\mathcal{G}_{ab}(x)$ and not on $\mathcal{Y}_b^a(x)$

3) Invariant under spatial coordinate transformations

Functional integral representation

HELP from a single degree of freedom

Every functional representative

$$\psi(p,\gamma) = \langle p, \gamma | \Psi \rangle$$

is analytic in $\gamma + (i/\beta)p$ up to a factor:

$$B\Psi(p,\gamma) = [-ig^{-1}\omega_p + 1 + \beta^{-1}g\partial_g] \Psi(p,\gamma) = 0$$

(called a "polarization")

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Let $A \equiv \frac{1}{2}\beta B^T B \geq 0$, then $A\psi(p, g) = 0$

$$\lim_{\nu \rightarrow \infty} (e^{-\nu TA}) \delta(p-p') \delta(g-g') = \Pi(p, g; p', g')$$

a projection kernel on zero subspace of A

But $\langle p, g | p', g' \rangle$ is proportional to this same kernel, therefore $\Pi \propto \langle p, g | p', g' \rangle$

Invoke Feynman-Kac-Stratonovich representation to learn that

$$\begin{aligned} \langle p'', g'' | p', g' \rangle &= \lim_{\nu \rightarrow \infty} \mathcal{M}_\nu \int e^{-i \int g \dot{p} dt - \frac{i}{2\nu} \int [\beta^{-1} g^2 \dot{p}^2 + \beta g^{-2} \dot{g}^2] dt} \\ &= \lim_{\nu \rightarrow \infty} 2\pi [1 - \frac{1}{2\beta}] e^{\nu T_2} \int e^{-i \int g dp} dW^\nu(p, g) \end{aligned}$$

Gravity case

$$\langle \pi'', g'' | \pi', g' \rangle$$

$$\begin{aligned} &= \lim_{\nu \rightarrow \infty} \mathcal{M}_\nu \int e^{-i/\nu \int g_{ab} \dot{\Pi}^{ab} d^3x dt} \\ &\times e^{-\frac{1}{2\nu h} \int [b(x)^\dagger g_{bc} g_{da} \dot{\Pi}^{ab} \dot{\Pi}^{cd} + b(x) g^{bc} g^{ad} \dot{g}_{ab} \dot{g}_{cd}] d^3x dt} \\ &\times \left[\Pi_{x,t} \Pi_{a,b} d\Pi^{ab}(x,t) dg_{ab}(x,t) \right] \end{aligned}$$

Integration domain limited to $\{g_{ab}(x,t)\} > 0$

Note similarities with single degree of freedom case!

Quantum constraints

$H_a(x) \rightarrow \mathcal{H}_a(x)$ diffeomorphism constraints

$H(x) \rightarrow \mathcal{H}(x)$ temporal constraint

Ideally (Dirac):

$$\mathcal{H}_a(x)|\Psi_{\text{phys}}\rangle = 0, \quad \mathcal{H}(x)|\Psi_{\text{phys}}\rangle = 0$$

Constraint commutators:

$$[\mathcal{H}_a(x), \mathcal{H}_b(y)] = i\hbar [\delta_{ab}(x,y) \mathcal{H}_b(x) - \delta_{ba}(x,y) \mathcal{H}_a(x)]$$

$$[\mathcal{H}_a(x), \mathcal{H}(y)] = i\hbar \delta_{ab}(x,y) \mathcal{H}(x)$$

$$[\mathcal{H}(x), \mathcal{H}(y)] = \frac{i\hbar}{2} \delta_{ab}(x,y) [\hat{g}^{ab}(x) \mathcal{H}_b(x) + \mathcal{H}_b(x) \hat{g}^{ab}(x)]$$

The fact that (generally)

$$\hat{g}^{ab}(x) |\Psi_{\text{phys}}\rangle \notin \mathcal{H}_{\text{phys}}$$

means that the constraints have become partially second class (c.f., $Q|\Psi_p\rangle = 0, P|\Psi_p\rangle = 0$, thus $[Q, P]|\Psi_p\rangle = i\hbar|\Psi_p\rangle = 0$, i.e. $\therefore |\Psi_p\rangle = 0$)

This is called an "anomaly". At this point most workers "change the theory"!

Projection operator method has no fear of second class constraints, and accepts the quantum constraints as given.

Introduce complete set of "orthonormal" functions $\{h_n(x)\}_{n=0}^{\infty}$ such that

$$\int h_n(x) h_m(x) b(x) d^3x = \delta_{nm}$$

$$b(x) \sum_{n=0}^{\infty} h_n(x) h_n(y) = \delta(x, y)$$

$$H_a(x) \rightarrow H_{a(n)} \equiv \int h_n(x) H_a(x) d^3x$$

$$H(x) \rightarrow H_{(n)} \equiv \int h_n(x) H(x) d^3x$$

$$\left\langle \sum_a \tilde{E}_a^2 \right\rangle = \sum_{n=0}^{\infty} 2^{-n} \left\{ H_{(n)}^2 + \sum_a H_{a(n)}^2 \right\}$$

Can arrange a suitable $R(N^a; N)$ so that

$$E \left(\sum_{n=0}^{\infty} 2^{-n} \left\{ H_{(n)}^2 + \sum_a H_{a(n)}^2 \right\} \leq \delta^2(\eta) \right)$$

$$= \int \prod_{a,t} e^{-i \int [N(x,t) H_a(x) + N(x,t) H(x)] d^3x dt} dR(N^a; N)$$

Insert into previous functional integral

$$\langle \pi^a g^a | E | \pi^c g^c \rangle$$

$$= \lim_{N \rightarrow \infty} \mathcal{M}_N \int e^{-i \int [g_{ab} \dot{\pi}^{ab} + N^a H_a^{(s)} + NH^{(s)}] d^3x dt}$$

$$\times e^{-i \int [b(x)^{-1} g_{bc} g_{da} \dot{\pi}^{ab} \dot{\pi}^{cd} + b(x) g^{bc} g^{da} \dot{g}_{ab} \dot{g}_{cd}] d^3x dt}$$

$$\times \left[\prod_{x,t} \prod_{a,b} d\pi^{ab}(x,t) dg_{ab}(x,t) \right] dR(N^a; N)$$

N.B., $H_a^{(s)}(x)$ & $H^{(s)}(x)$ are symbols for $H_a(x)$ & $H(x)$, i.e., "t-augmented" functions

Ideally,

$$\lim_{t \rightarrow 0} H_a^{(s)}(x) = H_a(x)$$

$$\lim_{t \rightarrow 0} H^{(s)}(x) = H(x)$$

However, need to know what $H_a(x)$ and $H(x)$ are — including t -dependent "counterterms" — to determine $H_a^{(s)}(x)$ and $H^{(s)}(x)$.

Counterterms for quantum gravity

Adopt the hard-core picture of nonrenormalizability

- Focus on scalar models

$$S_0(\hbar) = \eta_0 \int e^{\int \hbar \varphi d^3x - W_0(\varphi)} \mathcal{D}\varphi$$

$$S_\lambda(\hbar) = \eta_0 \int e^{\int \hbar \varphi d^3x - W_0(\varphi) - \lambda V(\varphi)} \mathcal{D}\varphi, \lambda > 0$$

introduce

$$X(\varphi) \equiv \begin{cases} 1, & W_0(\varphi) < \infty, V(\varphi) < \infty \\ 0, & W_0(\varphi) < \infty, V(\varphi) = \infty \end{cases}$$

$$S_\lambda(\hbar) = \eta_\lambda \int e^{\int \hbar \varphi d^3x - W_0(\varphi) - \lambda V(\varphi)} X(\varphi) \mathcal{D}\varphi, \lambda > 0$$

Renormalizable

$$X(\varphi) \equiv 1$$

$$S_\lambda(\hbar) \xrightarrow[\lambda \rightarrow 0]{} S_0(\hbar)$$

Nonrenormalizable

$$X(\varphi) \not\equiv 1$$

$$S_\lambda(\hbar) \xrightarrow[\lambda \rightarrow 0]{} S'_0(\hbar) \neq S_0(\hbar)$$

Can argue that $X(\text{grav}) \not\equiv 1$ as well!

What is known about \mathcal{G}_5^4 ?

$$\eta_\lambda \int e^{\int h\varphi d^5x} - \frac{1}{2} \int [(\nabla\varphi)^2 + m^2\varphi^2] d^5x - \lambda \int \varphi^4 d^5x$$

- perturbation theory



$$\propto \frac{p^{2.5}}{p^{2.5}} \rightarrow \ln(1) \text{ needs } \varphi^6 \text{ term}$$



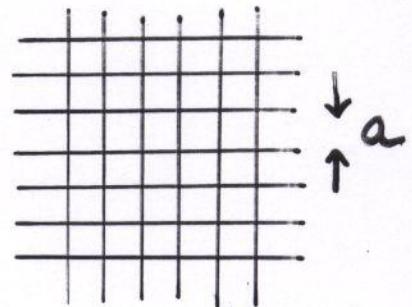
$$\propto \frac{p^{2.5}}{p^{2.4}} \rightarrow \lambda^2 \text{ needs } \varphi^8 \text{ term}$$

etc.; an infinite number of counterterms

- lattice regularization

$$N \iint e^{\sum h_h \varphi_h a^5 - \frac{1}{2} \sum (\varphi_{h+r} - \varphi_h)^2 a^3 - \frac{1}{2} m_0^2(a) \sum \varphi_h^2 a^5 - \lambda_0(a) \sum \varphi_h^4 a^5} \prod d\varphi_h$$

$$\underset{a \rightarrow 0}{\longrightarrow} e^{-\frac{1}{2} \int h(x) U(x-y) h(y) d^5x d^5y}$$



Hard core picture? YES

$$\int [(\nabla\varphi)^2 + m^2\varphi^2] d^5x < \infty$$

$$\int \varphi^4 d^5x = \infty$$

e.g., whenever $\varphi = \frac{e^{-|x|^p}}{|x|^p}, 1.25 < p < 1.5$

- Expansion of hard core leads to series of increasingly divergent contributions
- Lattice form lacks contribution from $\chi(\varphi)$

Can models with hard-core interactions be solved? YES (some)

$$1) \eta \int e^{i\hbar\varphi d^n x} - \frac{1}{2} m_0^2 \int \varphi^2 d^n x - \lambda_0 \int \varphi^4 d^n x \quad \text{d}\varphi$$

$$= \exp \left(- \frac{1}{2} b \int d^n x \left\{ 1 - \cos[u h(x)] \right\} e^{-\frac{1}{2} b m^2 u^2 - \lambda b^3 u^4} du / m \right)$$

where $m_0^2 = b m^2 / \delta(0)$, $\lambda_0 = b^3 \lambda / \delta(0)^3$

N.B. Limit as $\lambda_0 \rightarrow 0$ is not the "free" case!

$$2) \eta \int e^{i\hbar\varphi d^n x} - \frac{1}{2} \int [\dot{\varphi}^2 + m^2 \varphi^2] d^n x - \lambda_0 \int \varphi^4 d^n x \quad \text{d}\varphi$$

is soluble (reduced to quadrature)

For details see

"Beyond Conventional Quantization"
(Cambridge, 2000 & 2005)

Again, limit as $\lambda_0 \rightarrow 0$ is not the "free" case!

Lesson to be drawn

Hard-core picture has merit, and quite possibly may help understand and overcome perturbative nonrenormalizability of relativistic scalar theories, and, perhaps, of gravity itself.

