

SELECTED TOPICS

IN

QUANTUM THEORY

1. COHERENT STATES
2. PATH INTEGRALS
3. QUANTUM CONSTRAINTS
4. QUANTUM GRAVITY

by

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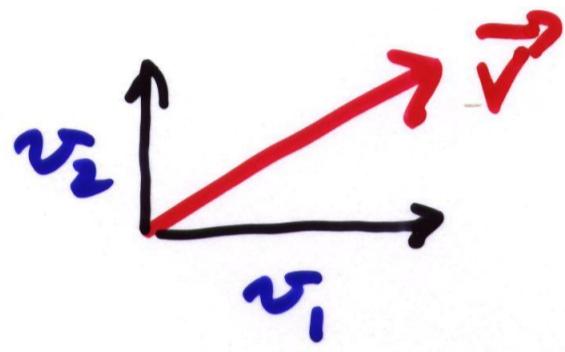
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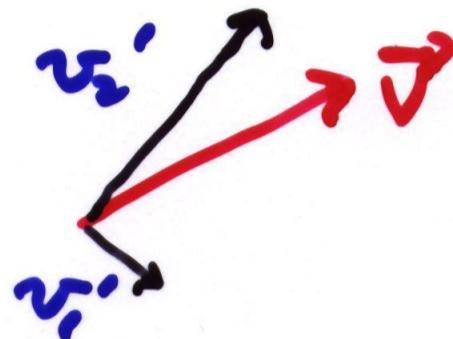
NORWEGIAN UNIVERSITY OF
SCIENCE AND TECHNOLOGY

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Vectors & Coordinates



$$\vec{v} = (v_1, v_2)$$



$$\vec{v}' = (v_1', v_2')$$

$$\vec{v} \cdot \vec{v} = v_1^2 + v_2^2 = v_1'^2 + v_2'^2$$

$$\vec{v} \rightarrow |v\rangle$$

$$\vec{e}_j \cdot \vec{v} \rightarrow \langle e_j | v \rangle = v_j$$

$$\vec{e}_j' \cdot \vec{v} \rightarrow \langle e_j' | v \rangle = v_j'$$

1. COHERENT STATES

First and foremost used for representations

Analogs

Discrete basis:

$$\{|n\rangle\}_{n=0}^{\infty}, \sum_{n=0}^{\infty} |n\rangle \langle n| = I, \quad \langle n|m\rangle = \delta_{nm}$$

$$|\psi\rangle \in \mathcal{H} \rightarrow \langle n|\psi\rangle = \psi_n \in \ell^2$$

$$\langle \varphi|\psi\rangle = \langle \varphi|I|\psi\rangle = \sum_n \langle \varphi|n\rangle \langle n|\psi\rangle = \sum_n \varphi_n^* \psi_n$$

$$B \in \mathcal{O} \rightarrow \langle n|B|m\rangle = B_{nm}; N|n\rangle = n|n\rangle$$

Continuous basis:

$$\{|x\rangle\}_{x \in \mathbb{R}}, \int |x\rangle \langle x| dx = I, \quad \langle x|\psi\rangle = \delta(x-y)$$

$$|\psi\rangle \in \mathcal{H} \rightarrow \langle x|\psi\rangle = \psi(x) \in L^2(\mathbb{R})$$

$$\langle \varphi|\psi\rangle = \int \langle \varphi|x\rangle \langle x|\psi\rangle dx = \int \varphi(x)^* \psi(x) dx$$

$$B \in \mathcal{O} \rightarrow \langle x|B|y\rangle = B(x, y); Q|x\rangle = x|x\rangle$$

Canonical coherent states:

$$\{|z\rangle\}_{z \in \mathbb{C}}, \int |z\rangle \langle z| d\mu(z) = I,$$

$$\langle z|z'\rangle = e^{-|z|^2/2 + z^* z' - |z'|^2/2}$$

$$d\mu(z) = \frac{d\text{Re}(z)}{\pi} \frac{d\text{Im}(z)}{\pi} = \frac{dx dy}{\pi}$$

$$|\psi\rangle \in \mathcal{H} \rightarrow \langle z|\psi\rangle = \psi(z) \in L^2(\mathbb{R}^2)$$

$$\langle \varphi|\psi\rangle = \int \langle \varphi|z\rangle \langle z|\psi\rangle d\mu = \int \varphi(z)^* \psi(z) d\mu$$

$$B \in \mathcal{O} \rightarrow \langle z|B|z'\rangle; a|z\rangle = z|z\rangle$$

$$[a, a^\dagger] = aa^\dagger - a^\dagger a = I ; N = a^\dagger a$$

1-2

$$N|n\rangle = n|n\rangle , \quad n=0, 1, 2, \dots$$

$$a|n\rangle = \sqrt{n}|n-1\rangle , \quad a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$$

$$|z\rangle = e^{-|z|^2/2} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} |n\rangle ; a|z\rangle = z|z\rangle$$

$$\langle z| = e^{-|z|^2/2} \sum_{n=0}^{\infty} \frac{z^{*n}}{\sqrt{n!}} \langle n| ; \langle z| a^\dagger = z^* \langle z|$$

Resolution of unity

$$z = r e^{i\theta}$$

$$\int |z\rangle \langle z| \frac{dx dy}{\pi} = \sum_{n,m} \frac{|n\rangle \langle m|}{\sqrt{n!m!}} \left\{ e^{-|z|^2} z^n z^{*m} dx dy / \pi \right.$$

$$= \sum_n \frac{|n\rangle \langle n|}{n!} \int e^{-r^2} r^{2n} 2r dr = \sum |n\rangle \langle n| = I$$

Normal ordering : Use $a a^\dagger = I + a^\dagger a$

$$B(a^\dagger, a) = :C(a^\dagger, a):$$

$$\langle z| B(a^\dagger, a) |z'\rangle = \langle z| :C(a^\dagger, a): |z'\rangle$$

$$= C(z^*, z') \langle z| z'\rangle = B(z, z')$$

Diagonal coherent state matrix elements

$$\begin{aligned} \langle z| \sum C_{mn} a^{*m} a^n |z\rangle &= \sum C_{mn} z^{*m} z^n \\ &= \sum C_{mn} r^{m+n} e^{i(n-m)\theta} \end{aligned}$$

$$\langle z| B |z'\rangle \iff \langle z| B |z\rangle = B(z)$$

Diagonal representation of operators

$$\begin{aligned} \sum D_{mn} a^m a^{*n} &= \sum D_{mn} \int a^m |z\rangle \langle z| a^{*n} d\mu \\ &= \int (\sum D_{mn} z^m z^{*n}) |z\rangle \langle z| d\mu \end{aligned}$$

(anti) normal-ordered symbols

Bargmann Space (Segal-Bargmann)

$|\psi\rangle \rightarrow f(z)$ entire function

$$\langle\varphi|\psi\rangle \rightarrow \int g(z)^* f(z) e^{-|z|^2} dx dy / \pi$$

$$a \rightarrow \frac{d}{dz}, \quad a^\dagger \rightarrow z$$

WHY

Harmonic oscillator Hamiltonian

$$\mathcal{H} = \hbar\omega(a^\dagger a + \frac{1}{2}) \rightarrow \hbar\omega(z \frac{d}{dz} + \frac{1}{2})$$

$$\hbar\omega(z \frac{d}{dz} + \frac{1}{2}) f_n(z) = E_n f_n(z)$$

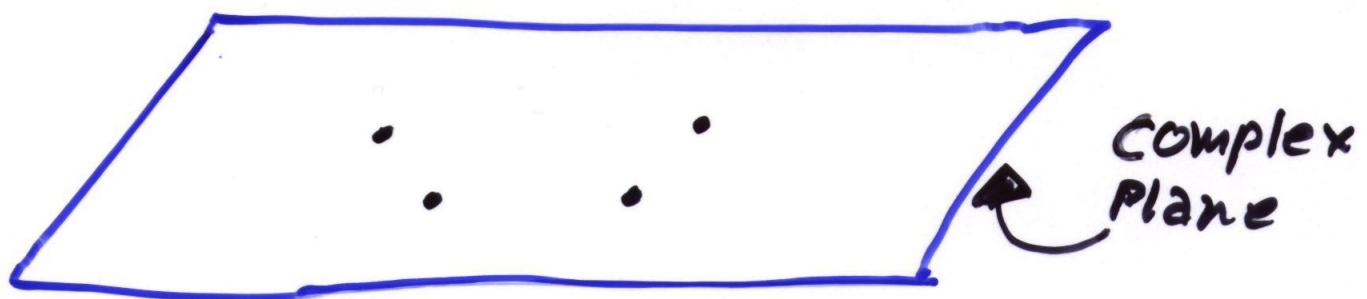
$$z \frac{d}{dz} f_n = \left(\frac{E_n}{\hbar\omega} - \frac{1}{2} \right) f_n$$

$$f_n \propto z^{\left(\frac{E_n}{\hbar\omega} - \frac{1}{2}\right)} = z^n, \quad n=0,1,2,\dots$$

$$\therefore E_n = \hbar\omega(n + \frac{1}{2})$$

Representation of $f(z)$ by its zeros

$$f(z) = (z - z_1)(z - z_2) \cdots (z - z_p)$$



A few complex numbers specify $f(z)$

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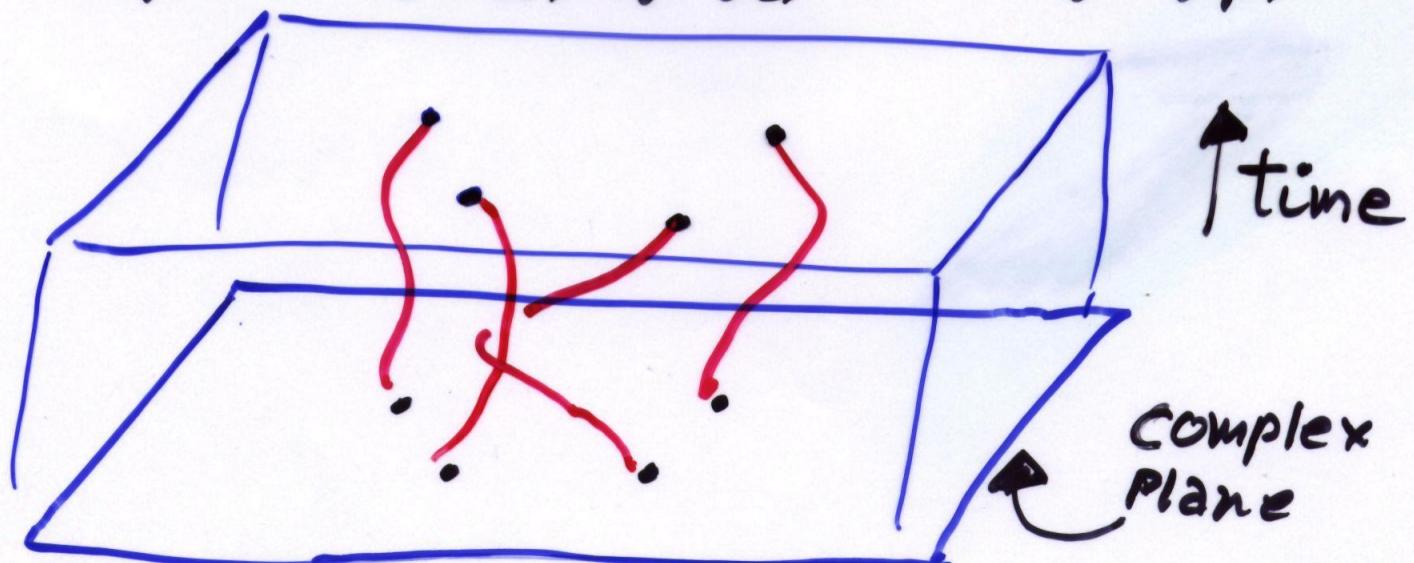
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Representation of $f(z)$ by its zeros

$$f(z) = (z - z_1)(z - z_2) \dots (z - z_p)$$



A few complex numbers specify $f(z)$

Connection with Phase Space

$$\begin{aligned}
 |\beta\rangle &= e^{(\beta a^\dagger - \beta^* a)} |0\rangle \\
 &= e^{-\frac{1}{2}|\beta|^2} e^{\beta a^\dagger} e^{-\beta^* a} |0\rangle \quad \boxed{\text{WHY}} \\
 &= e^{-\frac{1}{2}|\beta|^2} e^{\beta a^\dagger} |0\rangle, \quad a|0\rangle = 0 \\
 &= e^{-\frac{1}{2}|\beta|^2} \sum_{n=0}^{\infty} \frac{\beta^n}{n!} a^{+n} |0\rangle \\
 &= e^{-\frac{1}{2}|\beta|^2} \sum_{n=0}^{\infty} \frac{\beta^n}{\sqrt{n!}} |n\rangle \\
 &\dots \dots \dots \dots \dots \dots \dots \dots \\
 \beta &= \frac{g + iP}{\sqrt{2\pi}}, \quad a = \frac{Q + iP}{\sqrt{2\pi}}
 \end{aligned}$$

$$\begin{aligned}
 e^{\frac{i}{\hbar}[g(PQ - gP) - (g - iP)(Q + iP)]} \\
 = e^{i\hbar(PQ - gP)} = e^{(ga^\dagger - g^* a)}
 \end{aligned}$$

Weyl operator

Generalized canonical coherent states

$$\left\{
 \begin{aligned}
 |P, g; \eta\rangle &\equiv e^{i\hbar(PQ - gP)} |\eta\rangle = |P, g\rangle \\
 |P, g; \eta\rangle &\equiv e^{-i\hbar gP} e^{i\hbar PQ} |\eta\rangle = |P, g\rangle
 \end{aligned}
 \right.$$

differ by a phase

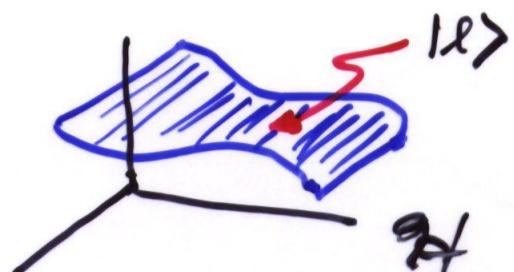
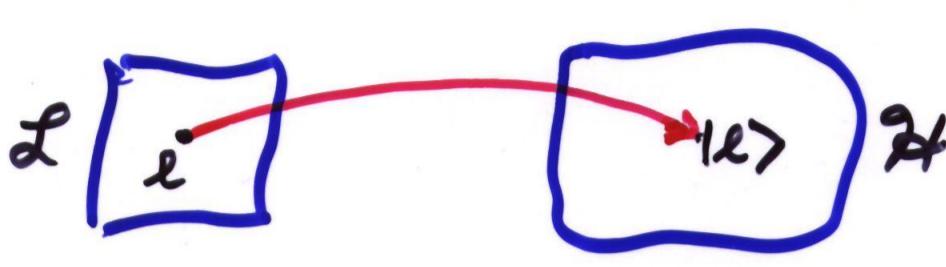
WHAT

Resolution of unity

$$\int |P, g\rangle \langle P, g| \frac{dp dg}{2\pi\hbar} = I = I \langle \eta | \eta \rangle$$

General definition of coherent states

$$\ell \in \mathcal{L} \approx \mathbb{R}^n, \quad |\ell\rangle \in \mathcal{H}$$



1) continuous map: $\ell' \rightarrow \ell \rightarrow |\ell'\rangle \rightarrow |\ell\rangle$
 i.e., $\langle \ell' | \ell \rangle$ jointly cont., or $\| |\ell'\rangle - |\ell\rangle \| \rightarrow 0$

2) positive measure: $\mu(\ell)$, $d\mu(\ell) = P(\ell) d^n \ell$

$$\int |\ell\rangle \langle \ell| d\mu(\ell) = I$$

Remarks:

- i) $|\ell\rangle \neq 0, P(\ell) > 0$ (a.e.)
- ii) can normalize so $\langle \ell | \ell \rangle = \| |\ell\rangle \| ^2 = 1$
- iii) groups are useful but not required

Group examples:

• Rotation group (spin s , $s=\frac{1}{2}, 1, \frac{3}{2} \dots$)

$$[S_j, S_k] = i \epsilon_{jkl} S_l; \quad \sum_s S_s^2 = s(s+1) I$$

$$|s\rangle = |s\rangle, \quad S_3 |s\rangle = s |s\rangle$$

$$|\theta, \phi\rangle \equiv e^{-i\varphi S_3} e^{-i\theta S_2} |s\rangle, \quad \| |\theta, \phi\rangle \| = 1$$

$$I = \int |\theta, \phi\rangle \langle \theta, \phi| d\mu(\theta, \phi)$$

$$= \frac{2s+1}{4\pi} \int |\theta, \phi\rangle \langle \theta, \phi| \sin \theta d\theta d\phi$$

WHY

$$|\psi\rangle \in \mathcal{H} \rightarrow \langle \theta, \varphi | \psi \rangle = \psi(\theta, \varphi) \in L^2(S^2)$$

$$B \in \mathcal{O} \rightarrow \langle \theta, \varphi | B | \theta', \varphi' \rangle$$

$$\langle \theta, \varphi | B | \theta', \varphi' \rangle \iff \langle \theta, \varphi | B | \theta, \varphi \rangle = B(\theta, \varphi)$$

$$B = \int b(\theta, \varphi) | \theta, \varphi \rangle \langle \theta, \varphi | d\mu(\theta, \varphi)$$

$$\langle \theta, \varphi | B | \theta', \varphi' \rangle = \left[\cos \frac{\theta}{2} \cos \frac{\theta'}{2} e^{-\frac{i}{2}(\varphi - \varphi')} + \sin \frac{\theta}{2} \sin \frac{\theta'}{2} e^{i(\varphi - \varphi')} \right]$$

• Affine group (wavelets)

$$[Q, P] = i\hbar I, \quad Q > 0$$

$$e^{i\beta P/\hbar} Q e^{-i\beta P/\hbar} = Q + \beta, \quad Q > 0 \text{ violated } \frown \text{ smiley}$$

$\therefore P$ is not observable

$$[Q, P] \cdot Q = i\hbar Q = [Q, D], \quad D = \frac{1}{2}(PQ + QP)$$

$$e^{i\alpha D/\hbar} Q e^{-i\alpha D/\hbar} = e^\alpha Q, \quad Q > 0 \text{ preserved } \smile$$

$$\text{choose } [Q, D] = i\hbar G, \quad Q > 0$$

$$\langle n | Q | n \rangle = 1, \quad \langle n | D | n \rangle = 0$$

$$[Q - 1 + i\beta^{-1}\hbar^{-1}D] |n\rangle = 0$$

$$|P, g\rangle \equiv e^{iPQ/\hbar} e^{-i\hbar \ln g D} |n\rangle$$

$$I = \int |P, g\rangle \langle P, g| d\mu(P, g)$$

$$d\mu(P, g) = \frac{dp dg}{2\pi\hbar C}, \quad C = \langle n | Q^{-1} | n \rangle < \infty$$

$(\beta > 1/2)$

$$|\psi\rangle \in \mathcal{H} \rightarrow \langle p, g | \psi \rangle = \psi(p, g) \in L^2(\mathbb{R} \times \mathbb{R}^+)$$

$$B \in \mathcal{O} \rightarrow \langle p, g | B | p', g' \rangle$$

$$\langle p, g | B | p', g' \rangle \Leftrightarrow \langle p, g | B | p, g \rangle = B(p, g)$$

$$B = \int b(p, g) |p, g\rangle \times_{p, g} d\mu(p, g)$$

$$\langle p, g | p', g' \rangle = \left\{ \frac{g^{-k_2} g'^{-k_2}}{\left[\frac{1}{2}(g^{-1} + g'^{-1}) + \frac{i}{2\beta k} (p - p') \right]} \right\}^{2\beta}$$

Non-group related example

$$|z\rangle \equiv N \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{P_n}} |n\rangle, \quad N^{-2} = \sum_{n=0}^{\infty} \frac{|z|^{2n}}{P_n}, \quad |z| < U$$

$$I = \int_{|z| < U} |z\rangle \times_z |d\mu(z)|, \quad d\mu = W(|z|) \frac{dx dy}{\pi}$$

$$\begin{aligned} &= \sum_{n=0}^{\infty} \frac{|n\rangle \times_n|}{P_n} \int N(|z|) W(|z|) |z|^{2n} d|z|^2 \\ &= \sum_{n=0}^{\infty} |n\rangle \times_n| = I \end{aligned}$$

Example:

$$N^2 W = \frac{1}{2} e^{-|z|}, \quad P_n = (2n+1)!$$

$$|z\rangle = \sqrt{\frac{1}{\sinh(|z|)}} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{(2n+1)!}} |n\rangle, \quad U = \infty$$

$$I = \int |z\rangle \times_z \left\{ \frac{(1 - e^{-2|z|})}{4|z|} \right\} \frac{dx dy}{\pi}$$

CHECK

Physics of coherent states: Classical & quantum mechanics

$$Q|x\rangle = x|x\rangle, \quad \dim[x] = \dim[Q]$$

$$P|k\rangle = k|k\rangle, \quad \dim[k] = \dim[P]$$

$$U[P, g] \equiv e^{-igP/\hbar} e^{iPQ/\hbar}$$

$$U[P, g]^{\dagger} P U[P, g] = P + p$$

WHY

$$U[P, g]^{\dagger} Q U[P, g] = Q + q$$

$$|P, g\rangle = U[P, g]|\eta\rangle$$

$$\langle P, g | P | P, g \rangle = P + \langle \eta | P | \eta \rangle$$

$$\langle P, g | Q | P, g \rangle = g + \langle \eta | Q | \eta \rangle$$

Choose $|\eta\rangle$ to be "physically centered":

$$\langle \eta | P | \eta \rangle = 0 = \langle \eta | Q | \eta \rangle \quad \therefore$$

$$\boxed{\langle P, g | P | P, g \rangle = P, \quad \langle P, g | Q | P, g \rangle = g}$$

$\{P\}_g$ is the mean value of $\{P_Q\}$

$$H(P, g) \equiv \langle P, g | H(P, Q) | P, g \rangle = \langle \eta | H(P+P_g, Q+q) | \eta \rangle$$

$$= H(P, Q) + O(\hbar; P, Q)$$

$$\text{provided } \langle \eta | (P^2 + Q^2) | \eta \rangle = O(\hbar)$$

$H(P, g)$ is an " \hbar -augmented" classical Hamiltonian

Interpret $|P, \dot{q}\rangle$ as variables in classical Phase space \mathcal{M}



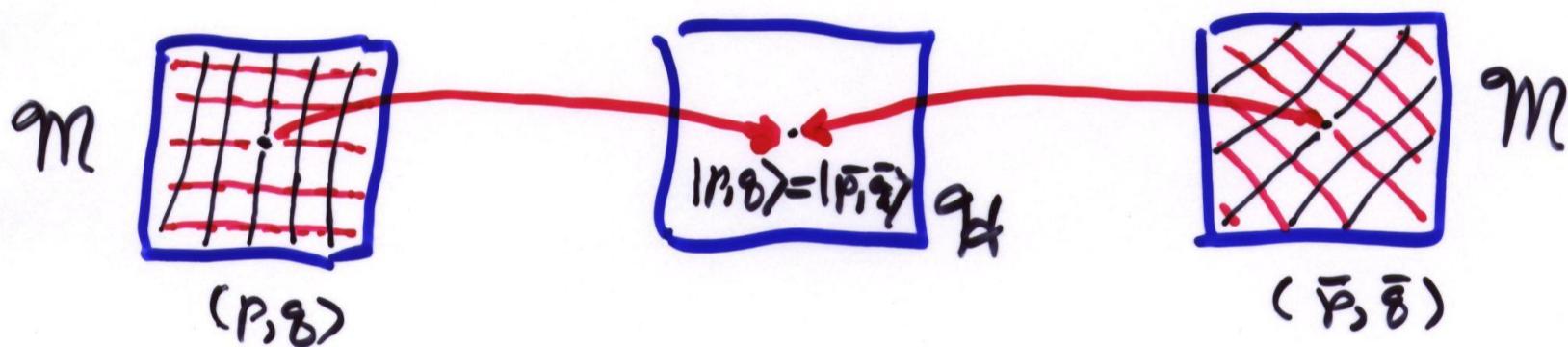
New canonical coordinates

$$\bar{P} = \bar{P}(P, \dot{q}), \quad \bar{q} = \bar{q}(P, \dot{q})$$

$$\begin{aligned} P d\dot{q} &= \bar{P} d\bar{\dot{q}} + d\bar{F}(\bar{q}, \dot{q}) \\ P d\dot{q} &= \bar{P} d\bar{\dot{q}} + d\bar{G}(\bar{P}, \bar{\dot{q}}) \end{aligned} \quad \left. \begin{array}{l} d\bar{P} d\bar{\dot{q}} \\ = dP d\dot{q} \end{array} \right\}$$

$$P = p(\bar{P}, \bar{\dot{q}}), \quad q = q(\bar{P}, \bar{\dot{q}})$$

$$|P, \dot{q}\rangle = |p(\bar{P}, \bar{\dot{q}}), q(\bar{P}, \bar{\dot{q}})\rangle \equiv |\bar{P}, \bar{\dot{q}}\rangle$$



Geometry of coherent states

- Symplectic geometry of $\{|P, \dot{q}\rangle\}$

$$\Theta \equiv i\hbar \langle P, \dot{q} | d | P, \dot{q} \rangle \quad (\text{one form})$$

$$i\hbar d(e^{-i\dot{q}P/\hbar} e^{iP\dot{Q}/\hbar}) |\eta\rangle$$

$$= [e^{-i\dot{q}P/\hbar} (P d\dot{q} - Q dP) e^{iP\dot{Q}/\hbar}] |\eta\rangle$$

$$\Theta = \langle \eta | e^{-ipQ/\hbar} (Pdg - Qdp) e^{ipQ/\hbar} | \eta \rangle$$

$$= \langle \eta | [(P + p)dg - Qdp] | \eta \rangle$$

$$= pdg + \langle \eta | P | \eta \rangle dg - \langle \eta | Q | \eta \rangle dp$$

choose $|\eta\rangle$ as physically centered

$$\Theta(p, g) = i\hbar \langle p, g | d | p, g \rangle = pdg$$

- Riemannian geometry of $\{|p, g\rangle\}$

Background

Let $|+\rangle$ and $|-\rangle$ be unit vectors

$$d^2 \equiv \| |+\rangle - |-\rangle \| ^2 = 2[1 - \text{Re} \langle +|-\rangle]$$

$$d_{\text{RAY}}^2 \equiv \min_{\alpha} \| |+\rangle - e^{i\alpha} |-\rangle \| ^2 = 2[1 - |\langle +|-\rangle|]$$

$$= 2 \left[1 - \left\{ \left[1 - \frac{1}{2} \| |+\rangle - |-\rangle \| ^2 \right]^2 + \frac{1}{4} |\langle +|-\rangle - \langle -|+\rangle|^2 \right\}^{\frac{1}{2}} \right]$$

specialize to $|+\rangle = |-\rangle + d|-\rangle$

$$\text{N.B. } \langle -|d|-\rangle^* = -\langle -|d|-\rangle$$

$$d_{\text{RAY}}^2 = \| d|-\rangle \| ^2 - |\langle -|d|-\rangle|^2$$

choose $|-\rangle = |p, g\rangle$ (multiply by $2\hbar$)

$$d\sigma^2(p, g) \equiv 2\hbar \left[\| d|Rg\rangle \| ^2 - |\langle Rg|d|Rg\rangle|^2 \right]$$

Notation: $\langle A \rangle \equiv \langle n | A | n \rangle$

Canonical coherent states:

$$|P, g\rangle = e^{-\frac{i}{\hbar} g P/\hbar} e^{i P Q / \hbar} |n\rangle$$

$$d\sigma^2 = \frac{2}{\hbar} [\langle Q^2 \rangle dP^2 + \langle PQ + QP \rangle dPdg + \langle P^2 \rangle dg^2]$$

Two-dimensional flat space for any $|n\rangle$

Harmonic oscillator ground state

$$|n\rangle = |0\rangle, (Q + iP)|0\rangle = 0$$

$$\boxed{d\sigma^2(P, g) = dP^2 + dg^2}$$

Cartesian coordinates

Geometry in different coordinates:

$$\mathbb{H}(\bar{P}, \bar{g}) = \bar{P} d\bar{g} + d\bar{E}(\bar{P}, \bar{g})$$

$$d\sigma^2(\bar{P}, \bar{g}) = A(\bar{P}, \bar{g}) d\bar{P}^2 + B(\bar{P}, \bar{g}) d\bar{P} d\bar{g} + C(\bar{P}, \bar{g}) d\bar{g}^2$$

still a flat phase space

Quantum mechanics gives a metric to classical phase space!

For canonical variables (i.e., kinematics), P and Q, the result is flat space

What about other quantum variables?

Spin coherent states

$$|\theta, \varphi\rangle = e^{-i\varphi S_3} e^{-i\theta S_2} |s\rangle$$

$$\Theta(\theta, \varphi) = i\hbar \langle \theta, \varphi | d | \theta, \varphi \rangle = s\hbar \cos \theta d\varphi$$

canonical variables

$$p = \sqrt{s\hbar} \cos \theta, \quad q = \sqrt{s\hbar} \varphi$$

$$\Theta(p, q) = pdq$$

$$d\sigma^2(\theta, \varphi) = s\hbar [d\theta^2 + \sin^2 \theta d\varphi^2]$$

$$d\sigma^2(p, q) = \frac{dp^2}{(1 - p^2/s\hbar)} + (1 - p^2/s\hbar) dq^2$$

sphere of radius $\sqrt{s\hbar}$, $s = \frac{1}{2}, 1, \frac{3}{2}, \dots$
(constant positive curvature)

Affine coherent states

$$|p, q\rangle = e^{ipq/\hbar} e^{-i\ln q D/\hbar} |n\rangle, \quad q > 0$$

$$\Theta(p, q) = i\hbar \langle p, q | d | p, q \rangle = pdq$$

$$d\sigma^2(p, q) = (\beta\hbar)^{-1} q^2 dp^2 + (\beta\hbar) q^{-2} dq^2$$

pseudo sphere (Poincaré plane)

(constant negative curvature)

Exercise:

Find Θ and $d\sigma^2$ for non-group $\{|z\rangle\}$

(leads to a non-constant curvature!)

Action principles of physics

- Classical action (phase space)

1) $I_c = \int_0^T [p\dot{q} - H(p, q)] dt$

$\delta I_c = 0$ holding $q(0)$ and $q(T)$ fixed

$$\delta I_c = \int \left[\left(\dot{q} - \frac{\partial H}{\partial p} \right) \delta p - \left(\dot{p} + \frac{\partial H}{\partial q} \right) \delta q \right] dt$$

$$+ p \delta q \Big|_0^T$$

2) $I_c = \int_0^T [-q\dot{p} - H(p, q)] dt$

$$\delta I_c = \int \left[\left(\dot{q} - \frac{\partial H}{\partial p} \right) \delta p - \left(\dot{p} + \frac{\partial H}{\partial q} \right) \delta q \right] dt$$

$$- q \delta p \Big|_0^T$$

3) $I_c = \int_0^T \left[\frac{1}{2}(p\dot{q} - q\dot{p}) - H(p, q) \right] dt$

$$\delta I_c = \int \left[\left(\dot{q} - \frac{\partial H}{\partial p} \right) \delta p - \left(\dot{p} + \frac{\partial H}{\partial q} \right) \delta q \right] dt$$

$$+ \frac{1}{2}(p \delta q - q \delta p) \Big|_0^T$$

4) $I_c = \int_0^T [p\dot{q} + \dot{G}(p, q) - H(p, q)] dt$

What values are held fixed at
the end points $t=0$ and $t=T$?

— Quantum action (Hilbert space)

$$I_Q = \int_0^T \left[\langle \psi | \{ i\hbar \frac{\partial}{\partial t} - H \} | \psi \rangle \right] dt$$

$$I_Q = \int_0^T \left\{ \left\{ \psi^*(x,t) i\hbar \frac{\partial \psi(x,t)}{\partial t} - \psi^*(x,t) H \psi(x,t) \right\} dx dt \right\}$$

$$\begin{aligned} \delta I_Q = & \iint \left[i\hbar \frac{\partial \psi}{\partial t} - H \psi \right] \delta \psi^* dx dt \\ & + \iint \left[-i\hbar \frac{\partial \psi^*}{\partial t} - H \psi^* \right] \delta \psi dx dt \\ & + i\hbar \int \psi^* \delta \psi dx \Big|_0^T \end{aligned}$$

What variables are held fixed here?

Two very different action principles of physics
— Maybe not!

Restricted variational principle

Restrict $|\psi(t)\rangle$ to $|p(t), g(t)\rangle$

$$\begin{aligned} I_Q|_{p,g} = & \int \left[i\hbar \langle p, g | \frac{d}{dt} | p, g \rangle - \langle p, g | H | p, g \rangle \right] dt \\ = & \int [p \dot{g} - H(p, g)] dt = I_C \end{aligned}$$

Classical action IS restricted form of quantum action!

We have called $H(p, q)$ "t-augmented"—does it make a difference?

Generally, correction terms are tiny and have only very small contributions.

But not always!

Example

Classical Hamiltonian

$$H_C = \frac{P^2}{2m} - \frac{e^2}{q}, \quad q > 0$$

(all solutions with $E < 0$ diverge)

Quantum Hamiltonian

$$H = \frac{P^2}{2m} - \frac{e^2}{Q}, \quad Q > 0$$

t-augmented classical Hamiltonian
(using affine coherent states)

$$\begin{aligned} H &= \langle p, q | \frac{P^2}{2m} - \frac{e^2}{Q} | p, q \rangle \\ &= \langle n | \frac{1}{2m} \left(\frac{P}{q} + p \right)^2 - \frac{e^2}{qQ} | n \rangle \\ &= \frac{\vec{p}^2}{2m} + \frac{\langle n | P^2 | n \rangle}{2m q^2} - \frac{e^2}{q} \langle n | Q | n \rangle \\ &\approx \frac{\vec{p}^2}{2m} - \frac{e^2}{q} + \frac{\hbar^2}{2me^2} \frac{1}{q} \frac{e^2}{q} \\ &= \frac{\vec{p}^2}{2m} - \frac{e^2}{q} + \left(\frac{a_0}{2q} \right) \frac{e^2}{q}, \quad a_0 = \frac{\text{Bohr radius}}{\text{radius}} \end{aligned}$$

All classical singularities removed!

2 PATH INTEGRALS

Schrödinger equation

$$i\hbar \frac{\partial \psi(x,t)}{\partial t} = \left\{ -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V(x) \right\} \psi(x,t)$$

$\psi(x,t)$, $t > 0$, determined by $\psi(x,0)$

$$\psi(x'',t) = \int K(x'',t; x',0) \psi(x',0) dx'$$

K is called the Propagator

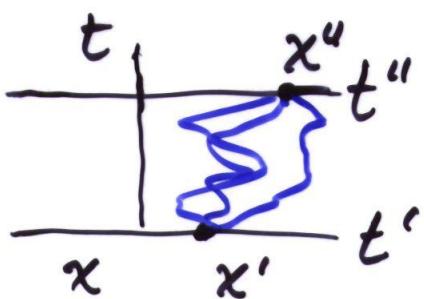
Solution of Schrödinger's equation
with initial condition: $\delta(x'' - x')$

Composition condition

$$K(x'',t''; x',t') \\ = \int K(x'',t''; x, t) K(x, t; x', t') dx$$

Feynman's 1st proposal (1948)

$$K(x'',t''; x',t') = \eta \int e^{i \frac{\hbar}{\mu} \int_{t'}^{t''} \left[\frac{m}{2} \dot{x}(t)^2 - V(x(t)) \right] dt} dx$$



classical action

transcendental concept



Define by "lattice regularization"

Lattice regularization $(t'' - t') = (N+1)\epsilon$ 2-2

$$K(x'', t''; x', t') \\ = \lim_{N \rightarrow \infty} N \int e^{\frac{i}{\hbar} \sum_{k=0}^N \left\{ \frac{m(x_{k+1} - x_k)^2}{2\epsilon} - \epsilon V(x_k) \right\}} \prod_{k=1}^N dx_k$$

plausible idea, but needs a proof:

works if $\frac{P^2}{2m} + V(Q)$ is essentially self adjoint on $D(P^2) \cap D(V)$

works for $V = Q^4$, but fails for $V = -Q^4$

Feynman-Kac formula (1950)

change $t \rightarrow -it$, $\hbar = 1$

$$\frac{\partial}{\partial t} P(x, t) = \left\{ \frac{1}{2m} \frac{\partial^2}{\partial x^2} - V(x) \right\} P(x, t)$$

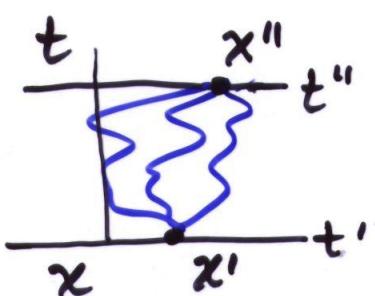
$$P(x'', t) = \int L(x'', t; x', 0) P(x', 0) dx'$$

$$L(x'', t''; x', t')$$

$$= q \int e^{- \int_{t'}^{t''} \left[\frac{m}{2} \dot{x}(t)^2 + V(x(t)) \right] dt} dx$$

$$= \lim_{N \rightarrow \infty} N \int e^{- \sum_{k=0}^N \left[\frac{m(x_{k+1} - x_k)^2}{2\epsilon} + \epsilon V(x_k) \right]} \prod_{k=1}^N dx_k$$

$$= \int e^{- \int_{t'}^{t''} V(x(t)) dt} d\mu_W(x)$$



$d\mu_W$ genuine (pinned) measure concentrated on continuous but nowhere differentiable paths

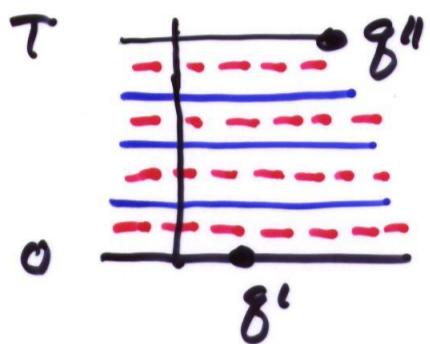
Feynman's 2nd proposal (1950)²⁻³

$$i\hbar \frac{\partial \psi(x,t)}{\partial t} = \mathcal{H}\left(-i\hbar \frac{\partial}{\partial x}, x\right) \psi(x,t)$$

$$\psi(g'', T) = \int K(g'', T; g', 0) \psi(g'; 0) dg'$$

$$K(g'', T; g'; 0) = \mathcal{M} \int e^{i\hbar \sum_t [p(t) g'(t) - H(p(t), g(t))] dt} \Omega p dg$$

$g'' = g(T)$, $g' = g(0)$; integrate over all p



lattice formulation

$$T = (N+1)\epsilon$$

$$K(g'', T; g', 0)$$

$$= \lim_{N \rightarrow \infty} \int \int e^{i\hbar \sum_{\ell=0}^N [p_{\ell+\frac{1}{2}}(g_{\ell+1} - g_\ell) - \epsilon H(p_{\ell+\frac{1}{2}}, g_\ell)]} \times \prod_{\ell=0}^N \frac{dp_{\ell+\frac{1}{2}}}{2\pi\hbar} \prod_{\ell=1}^N dg_\ell$$

N.B. One more p integration than g integration. Required by

$$K(g'', t''; g', t') = \int K(g''; t''; g, t) K(g, t; g', t') dg$$

$$H(p_{\ell+\frac{1}{2}}, g_\ell) = \frac{\langle p_{\ell+\frac{1}{2}} | \gamma_1 | g_\ell \rangle}{\langle p_{\ell+\frac{1}{2}} | g_\ell \rangle}$$

If $\mathcal{H} = \frac{P^2}{2m} + V(Q)$

then

$$H(p_{\ell+\frac{1}{2}}, g_\ell) = \frac{1}{2m} p_{\ell+\frac{1}{2}}^2 + V(g_\ell)$$

For such special Hamiltonians can integrate out p variables and recover original configuration space formulation

Proof of general expression ($\hbar=1$)

$$K(q^*, T; q^*, 0) = \langle q^* | e^{-iT\mathcal{H}} | q^* \rangle \\ = \langle q^* | e^{-i\varepsilon\mathcal{H}} e^{-i\varepsilon\mathcal{H}} \dots e^{-i\varepsilon\mathcal{H}} | q^* \rangle$$

$$= \int \dots \int_{\lambda=0}^N \langle q_{\lambda+1} | e^{-i\varepsilon\mathcal{H}} | q_\lambda \rangle \prod_{\lambda=1}^N dq_\lambda$$

$$= \int \dots \int_{\lambda=0}^N \langle q_{\lambda+1} | p_{\lambda+\frac{1}{2}} \rangle \langle p_{\lambda+\frac{1}{2}} | e^{-i\varepsilon\mathcal{H}} | q_\lambda \rangle \prod_{\lambda=0}^N dp_{\lambda+\frac{1}{2}} \prod_{\lambda=1}^N dq_\lambda$$

$$\left(\begin{aligned} \langle p | e^{-i\varepsilon\mathcal{H}} | q \rangle &\approx \langle p | [1 - i\varepsilon\mathcal{H}] | q \rangle \\ &= \langle p | q \rangle - i\varepsilon \langle p | \mathcal{H} | q \rangle \approx \langle p | q \rangle e^{-i\varepsilon \frac{\langle p | \mathcal{H} | q \rangle}{\langle p | q \rangle}} \end{aligned} \right)$$

$$= \lim_{N \rightarrow \infty} \int \dots \int_{\lambda=0}^N \langle q_{\lambda+1} | p_{\lambda+\frac{1}{2}} \times p_{\lambda+\frac{1}{2}} | q_\lambda \rangle e^{-i\varepsilon \langle p_{\lambda+\frac{1}{2}} | \mathcal{H} | q_\lambda \rangle / \langle p_{\lambda+\frac{1}{2}} | q_\lambda \rangle} \\ \times \prod_{\lambda=0}^N dp_{\lambda+\frac{1}{2}} \prod_{\lambda=1}^N dq_\lambda$$

$$= \lim_{N \rightarrow \infty} \int \dots \int e^{i\frac{q}{\hbar} \sum_{\lambda=0}^N [p_{\lambda+\frac{1}{2}} (q_{\lambda+1} - q_\lambda) - \varepsilon H(p_{\lambda+\frac{1}{2}}, q_\lambda)]} \\ \cdot \prod_{\lambda=0}^N \frac{dp_{\lambda+\frac{1}{2}}}{2\pi\hbar} \prod_{\lambda=1}^N dq_\lambda$$

Comments:

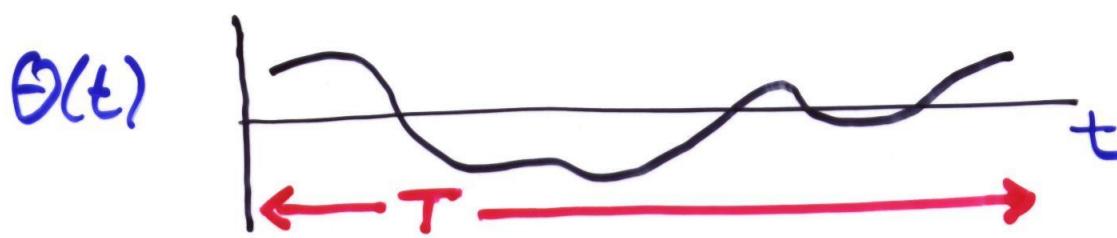
Meaning of $p + q$ as sharp eigenvalues

Alternate "knowing" q, p, q, p, q, \dots

Formulation implicitly states we know $p + q$ alternately

Unnatural (like "epicycles")

Epicycles model planetary motion



$$\Theta(t) = \sum_n a_n \sin(\varphi_n + 2\pi n t/T)$$

make a model to "explain" behavior
Can we do "better"?

Coherent state path integral (1960)

$$\Psi(p'', g'', T) = \langle p'', g'' | e^{-i\mathcal{H}T/\hbar} | \psi \rangle$$

$$= \int \langle p'', g'' | e^{-i\mathcal{H}T/\hbar} | p', g' \rangle \langle p', g' | \psi \rangle d\mu(p', g')$$

$$\Psi(p'', g'', T) = \int K(p'', g'', T; p', g', 0) \Psi(p', g'; 0) dp' dg' / 2\pi\hbar$$

$$\langle p'', g'' | e^{-i\mathcal{H}T} | p', g' \rangle = \langle p'', g'' | e^{-i\varepsilon\mathcal{H}} e^{-i\varepsilon\mathcal{H}} \dots e^{-i\varepsilon\mathcal{H}} | p', g' \rangle$$

$$(\hbar = 1 ; \quad T = (N+1)\varepsilon)$$

$$= \int \int \prod_{k=0}^N \langle p_{k+1}, g_{k+1} | e^{-i\varepsilon\mathcal{H}} | p_k, g_k \rangle \prod_{k=1}^N d\mu(p_k, g_k)$$

$$\langle p, g | e^{-i\varepsilon\mathcal{H}} | n, s \rangle \simeq \langle p, g | [1 - i\varepsilon\mathcal{H}] | n, s \rangle$$

$$= \langle p, g | n, s \rangle - i\varepsilon \langle p, g | \mathcal{H} | n, s \rangle$$

$$\simeq \langle p, g | n, s \rangle e^{-i\varepsilon H(p, g; n, s)}$$

$$\text{where } H(p, g; n, s) = \frac{\langle p, g | \mathcal{H} | n, s \rangle}{\langle p, g | n, s \rangle}$$

$$K(p'', g'', T; p', g', 0)$$

$$= \lim_{N \rightarrow \infty} \iint \prod_{\ell=0}^N \langle p_{\ell+1}, g_{\ell+1} | p_\ell, g_\ell \rangle e^{-i\varepsilon H(p_{\ell+1}, g_{\ell+1}; p_\ell, g_\ell)} \prod_{\ell=0}^N d\mu(p_\ell, g_\ell)$$

Interchange limit and \iint (formally!) and write integrand for continuous and differentiable paths (restore \hbar)

$$\text{Use } \langle z|l\rangle = 1 - \langle l(z)-l\rangle \approx e^{i\langle z|(lz)-l\rangle}$$

$$K(p'', g'', T; p', g', 0) = \mathcal{M} \int e^{i\hbar \int [i\hbar \langle p, g | d|p, g\rangle - \langle p, g | H | p, g \rangle] dt} \frac{dp dg}{\partial p \partial g}$$

$$= \mathcal{M} \int e^{i\hbar \int [p \dot{g} - H(p, g)] dt} \frac{dp dg}{\partial p \partial g}$$

Comments:

Meaning of $p + g$ as mean values

Simultaneous "knowing" of $p + g$ is O.K.

Natural formulation

Another formulation ($\hbar=1$)

$$I = \int |p, g \times p, g| d\mu(p, g)$$

$$H = \int \hbar(p, g) |p, g \times p, g| d\mu(p, g)$$

$$I - i\varepsilon H = \int [1 - i\varepsilon \hbar(p, g)] |p, g \times p, g| d\mu(p, g)$$

$$e^{-i\varepsilon H} \approx \int e^{-i\varepsilon \hbar(p, g)} |p, g \times p, g| d\mu(p, g)$$

$$\langle p'', g'' | e^{-iHT} | p', g' \rangle = \langle p'', g'' | e^{-i\varepsilon H} e^{-i\varepsilon H} \dots e^{-i\varepsilon H} | p', g' \rangle$$

$$= \lim_{N \rightarrow \infty} \int \int \prod_{k=0}^N \langle p_{k+1}, q_{k+1} | p_k, q_k \rangle \prod_{k=1}^N e^{-i\varepsilon h(p_k, q_k)} d\mu(p_k, q_k)$$

Again, interchange orders and write formal path integral for continuous and differentiable paths

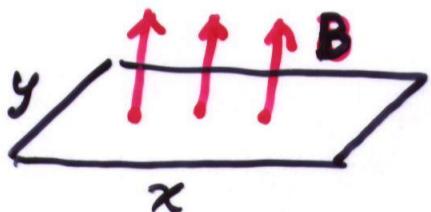
$$K(p'', q'', T; p', q', 0) = \mathcal{M} \int e^{i \int_0^T [i\dot{t} \langle p, q | d(p, q) - h(p, q) dt]} Dp Dq$$

$$= \mathcal{M} \int e^{i \int_0^T [p \dot{q} - h(p, q)] dt} Dp Dq$$

Involves a different symbol (!) — added proof of formal nature of continuum expression

Continuous-time regularization

Classical two-dimensional particle in a potential and a magnetic field



$$\underline{r} = (x, y, 0), \underline{B} = (0, 0, B)$$

$$\underline{A} = (0, Bx, 0), \nabla \times \underline{A} = \underline{B}$$

$$\nabla V = V(x, y)$$

Newton's equation

$$m \ddot{\underline{r}} = -\nabla V + \underline{r} \times \underline{B} \quad (e=c=1)$$

Now let $m \rightarrow 0$

$$\ddot{\underline{r}} \times \underline{B} = \nabla V \Rightarrow B \dot{y} = \frac{\partial V}{\partial x}$$

$$B \dot{x} = -\frac{\partial V}{\partial y}$$

$$\sqrt{B}x \rightarrow p$$

$$\sqrt{B}y \rightarrow q$$

$$V(x, y) \rightarrow H(p, q)$$

$$\dot{q} = \frac{\partial H}{\partial p}$$

$$\dot{p} = -\frac{\partial H}{\partial q}$$

Hamilton's equations

Quantum propagator of two-dim. System

$$\Psi(x'', y'', T) = \int K(x'', y'', T; x', y'; 0) \Psi(x', y'; 0) dx' dy'$$

$$K(x'', y'', T; x', y'; 0)$$

$$= \eta \int e^{i\hbar t} \left[\hat{A} \cdot \hat{r} + \frac{m}{2} \hat{r}^2 - V(\hat{r}) \right] dt \quad dxdy$$

Limit $m \rightarrow 0$, or $m \rightarrow i\infty \rightarrow 0$

$$= \lim_{m \rightarrow 0} \eta \int e^{i\hbar t} \left[\hat{A} \cdot \hat{r} - V(\hat{r}) \right] dt - \frac{m}{2\hbar} \int (\dot{x}^2 + \dot{y}^2) dt \quad dxdy$$

Change names: $\sqrt{B}x \rightarrow p$, $\sqrt{B}y \rightarrow q$, $V \rightarrow H$, $m = B/\nu$

$$K(p'', q'', T; p', q', 0)$$

$$= \lim_{\nu \rightarrow \infty} \eta \int e^{i\hbar t} \left[pq - H(p, q) \right] dt - \frac{i}{2\nu\hbar} \int (p^2 + q^2) dt \quad dp dq$$

Feynman-Kac-like expression

$$= \lim_{\nu \rightarrow \infty} 2\pi e^{\nu T/2} \int e^{i\hbar t} \left[pdq - H(p, q) dt \right] d\mu_w^\nu(p, q)$$

Well-defined path integral

$$K(p'', q'', T; p', q', 0) \equiv \langle p'', q'' | e^{-i\hbar H T / \hbar} | p', q' \rangle$$

$$|p, q\rangle \equiv e^{-iqP/\hbar} e^{ipQ/\hbar} |0\rangle$$

$$[Q, P] = i\hbar I, \quad (Q + iP)|0\rangle = 0$$

$$H = \int H(p, q) |p, q\rangle \langle p, q| dp dq / 2\pi\hbar$$

Solves Sch. equation for wide class of Hamiltonians

Covariance under coordinate transformations

$$K(p'', \dot{q}'', T; p', \dot{q}', 0) = \langle p'', \dot{q}'' | e^{-iT\mathcal{H}/\hbar} | p', \dot{q}' \rangle$$

$$= \lim_{N \rightarrow \infty} \frac{1}{2\pi\hbar} e^{iNT/2} \int e^{i\frac{\hbar t}{\hbar} \int [pd\dot{q} - H(p, q)dt]} d\mu_w^N(p, q)$$

$$\bar{K}(\bar{p}'', \bar{\dot{q}}'', T; \bar{p}', \bar{\dot{q}}', 0) = \langle \bar{p}'', \bar{\dot{q}}'' | e^{-iT\bar{\mathcal{H}}/\hbar} | \bar{p}', \bar{\dot{q}}' \rangle$$

$$= \lim_{N \rightarrow \infty} \frac{1}{2\pi\hbar} e^{iNT/2} \int e^{i\frac{\hbar t}{\hbar} \int [\bar{p}d\bar{\dot{q}} + d\bar{G}(\bar{p}, \bar{\dot{q}}) - \bar{H}(\bar{p}, \bar{\dot{q}})dt]} d\mu_w^N(\bar{p}, \bar{\dot{q}})$$

(uses Stratonovich (mid-point) rule)

$$\mathcal{H} = \int H(p, q) |p, q\rangle \langle p, q| dp dq / 2\pi\hbar$$

$$\bar{\mathcal{H}} = \int \bar{H}(\bar{p}, \bar{\dot{q}}) | \bar{p}, \bar{\dot{q}} \rangle \langle \bar{p}, \bar{\dot{q}} | d\bar{p} d\bar{\dot{q}} / 2\pi\hbar$$

$$d\mu_w^N(p, q) = \eta e^{-\frac{1}{2\hbar N} \int [p^2 + q^2] dt} dp dq$$

$$d\mu_w^N(\bar{p}, \bar{\dot{q}}) = \eta e^{-\frac{1}{2\hbar N} \int [A\bar{p}^2 + B\bar{p}\bar{\dot{q}} + C\bar{\dot{q}}^2] dt} d\bar{p} d\bar{\dot{q}}$$

Coordinate covariance is natural because the mathematics involves the simple transcription $(x, y) \rightarrow (p, q)$ of a genuine physical problem.

Conditionally convergent integral

$$\int_{-\infty}^{\infty} e^{iy^2/2} dy = \lim_{\substack{L_2 \rightarrow \infty \\ L_1 \rightarrow \infty}} \int_{-L_1}^{L_2} e^{iy^2/2} dy$$

$$= \lim_{N \rightarrow \infty} \int_{-\infty}^{\infty} e^{iy^2/2} e^{-y^2/2N} dy = \lim_{N \rightarrow \infty} \sqrt{\frac{2\pi}{1/N - i}} = \sqrt{2\pi i}$$

Convergence factors for path integrals

$$M \int e^{i/t \int [p\dot{q} - H(p, q)] dt} \frac{dp dq}{Dp Dq}$$

No mathematics!

No physics!

Regularize

$$\lim_{N \rightarrow \infty} M \int e^{i/t \int [p\dot{q} - H(p, q)] dt} e^{-\frac{1}{2Nt} \int (\dot{p}^2 + \dot{q}^2) dt} \frac{dp dq}{Dp Dq}$$

$$= \lim_{N \rightarrow \infty} 2\pi t e^{2T/2} \int e^{i/t \int [p\dot{q} - H(p, q) dt]} du_w(p, q)$$

$$= \langle p''; q'' | e^{-i T \hat{H}/t} | p, q' \rangle, \text{ canonical } [Q, P] = i\hbar I$$

Other geometries

$$\lim_{N \rightarrow \infty} M \int e^{i/t \int [p\dot{q} - H(p, q)] dt} e^{-\frac{1}{2Nt} \int \left[\frac{\dot{q}^2}{1-p^2(t)} + (1-p^2(t))\dot{p}^2 \right] dt} \frac{dp dq}{Dp Dq}$$

leads to spin kinematics: $[S_j, S_k] = i\varepsilon_{jkl} S_l$

$$\lim_{N \rightarrow \infty} M \int e^{i/t \int [p\dot{q} - H(p, q)] dt} e^{-\frac{1}{2Nt} \int [\beta^{-1} \dot{q}^2 \dot{p}^2 + \beta \dot{q}^{-2} \dot{q}^2] dt} \frac{dp dq}{Dp Dq}$$

leads to affine kinematics: $[Q, D] = i\hbar Q, Q > 0$

- Phase space metric gives both mathematical & physical meaning to formal path integral
- Leads to coherent state representation directly

Quantization IS Geometry, After All!

Evaluation of Integrals

$$F(y) = \frac{1}{2\pi} \int_0^{2\pi} e^{iy \cos \theta} d\theta$$

1) Power series: $F(y) = 1 - \frac{y^2}{2! \cdot 2} + \frac{3y^4}{4! \cdot 8} - \dots$

2) Differential equations: $yF''(y) + F'(y) + yF(y) = 0$

$$F(0) = 1, \quad F'(0) = 0$$

$\Rightarrow F(y) = J_0(y)$ (Bessel function)

Path integrals

$$\begin{aligned} K(x''; T; x') &= \mathcal{N} \int e^{i\hbar \int [\frac{m}{2} \dot{x}(t)^2 - V(x(t))] dt} dx \\ &= \langle x'' | e^{-iHt/\hbar} | x' \rangle \\ &= \sum_{n=0}^{\infty} \psi_n(x'') e^{-iE_n t/\hbar} \psi_n^*(x') \end{aligned}$$

Sometimes can be summed \Rightarrow Table of Path Integrals

Semiclassical Approximation of Path Integrals

Three representations

1) $\psi(x) = \langle x | \psi \rangle$

2) $\tilde{\psi}(p) = \langle p | \psi \rangle = \int \langle p | x \rangle \langle x | \psi \rangle dx$
 $= \frac{1}{\sqrt{2\pi\hbar}} \int e^{-ipx/\hbar} \psi(x) dx$

3) $\Psi(p, q) = \langle p, q | \psi \rangle = \int \langle p, q | x \rangle \langle x | \psi \rangle dx$
 $= \left(\frac{\Omega}{\pi\hbar}\right)^{1/4} \int e^{-\frac{\Omega(q-x)^2}{2\hbar}} e^{ip(q-x)/\hbar} \psi(x) dx$

N.B. $(\Omega Q + iP) |\eta \rangle = 0$

Additional connections

a) $\left(\frac{\pi\hbar}{\Omega}\right)^{1/4} \frac{1}{2\pi\hbar} \int \Psi(p, g) dp = \Psi(g)$

b1) $\lim_{\Omega \rightarrow \infty} \left(\frac{\Omega}{4\pi\hbar}\right)^{1/4} \Psi(p, g) = \Psi(g)$

b2) $\lim_{\Omega \rightarrow 0} \left(\frac{1}{4\pi\hbar\Omega}\right)^{1/4} \Psi(p, g) = e^{ipg/\hbar} \tilde{\Psi}(g)$

One basic identity

$$\begin{aligned} \langle x'' | e^{-iTH/\hbar} | x' \rangle &\equiv \int \langle x'' | p'' \rangle \langle p'' | e^{-iTH/\hbar} | x' \rangle dp'' \\ &\equiv \int \langle x'' | p'', g'' \rangle \langle p'', g'' | e^{-iTH/\hbar} | x' \rangle dp'' dg'' / 2\pi\hbar \end{aligned}$$

Three basic semiclassical ($= \infty$) approximations

$$\begin{aligned} [\langle x'' | e^{-iTH/\hbar} | x' \rangle]_{sc} &\approx \int \langle x'' | p'' \rangle [\langle p'' | e^{-iTH/\hbar} | x' \rangle]_{sc} dp'' \\ &\approx \int \langle x'' | p'', g'' \rangle [\langle p'', g'' | e^{-iTH/\hbar} | x' \rangle]_{sc} dp'' dg'' / 2\pi\hbar \end{aligned}$$

e.p. = extremal paths

$$[\langle x'' | e^{-iTH/\hbar} | x' \rangle]_{sc} = \sum_{e.p.} \frac{1}{\sqrt{2\pi\hbar [i\tilde{x}(\tau)]}} e^{iS_1(x''; x')/\hbar}$$

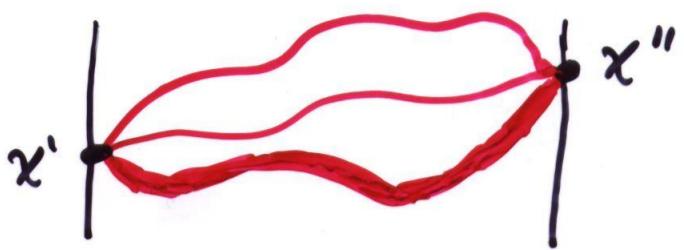
$$[\langle p'' | e^{-iTH/\hbar} | x' \rangle]_{sc} = \sum_{e.p.} \frac{1}{\sqrt{2\pi\hbar [\tilde{p}(\tau)]}} e^{iS_2(p''; x')/\hbar}$$

$$[\langle p'', g'' | e^{-iTH/\hbar} | x' \rangle]_{sc} = \frac{\sqrt{\Omega/\pi\hbar}}{\sqrt{[\tilde{p}(\tau) + i\Omega\tilde{x}(\tau)]}} e^{iS_3(p'', g''; x')/\hbar}$$

Multiple extremal paths

$$\dot{x} = \frac{\partial H}{\partial p} \equiv H_p(p, x)$$

$$\dot{p} = -\frac{\partial H}{\partial x} \equiv -H_x(p, x)$$



paths differentiated by initial / final slope

Linearized deviations about extremal paths

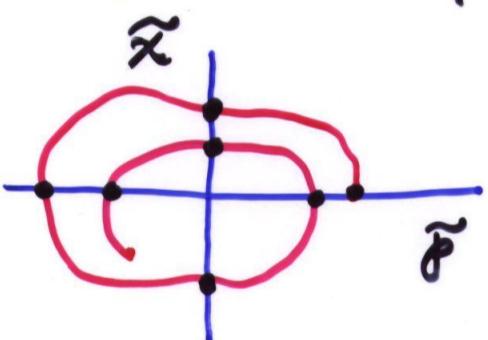
$$\dot{x} + \tilde{x} = H_p(p + \tilde{p}, x + \tilde{x}), \quad \dot{p} + \tilde{p} = -H_x(p + \tilde{p}, x + \tilde{x})$$

$$\dot{\tilde{x}} = H_{pp}(p, x) \tilde{p} + H_{px}(p, x) \tilde{x}$$

$$\dot{\tilde{p}} = -H_{xp}(p, x) \tilde{p} - H_{xx}(p, x) \tilde{x}$$

Solution

$$\begin{pmatrix} \tilde{x}(T) \\ \tilde{p}(T) \end{pmatrix} = \begin{pmatrix} A(T) & B(T) \\ C(T) & D(T) \end{pmatrix} \begin{pmatrix} \tilde{x}(0) = 0 \\ \tilde{p}(0) = 1 \end{pmatrix}$$



Meaning : multiple solutions

caustic : $\tilde{x}(T) = 0$

pseudocaustic : $\tilde{p}(T) = 0$

N.B. Can never have $\tilde{x}(T) = 0$ and $\tilde{p}(T) = 0$

Coherent state "se" approximation involves

$$\frac{1}{\sqrt{[\tilde{p}(T) + i\Omega \tilde{x}(T)]}}$$

and overcomes problem of multiple solutions

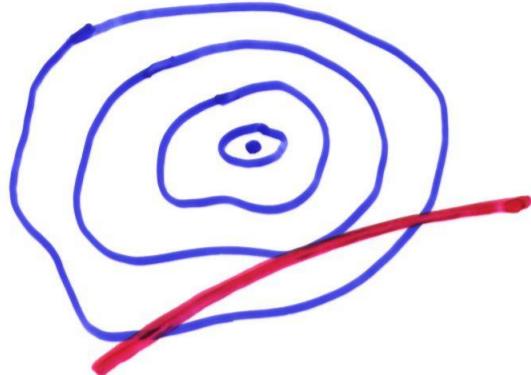
N.B. $\int dp$ incorporates multiple extremal paths

Used in oil exploration & in

under water detection

3. QUANTUM CONSTRAINTS

The Role of Lagrange Multipliers



height: $f(x, y)$

extremal: $\vec{\nabla}f(x, y) = 0$

constraint: $\varphi(x, y) = 0$

compatibility: $\vec{\nabla}f \propto \vec{\nabla}\varphi$

set of equations: $\vec{\nabla}f(x, y) = \lambda \vec{\nabla}\varphi(x, y), \varphi(x, y) = 0$

Classical Mechanics with Constraints

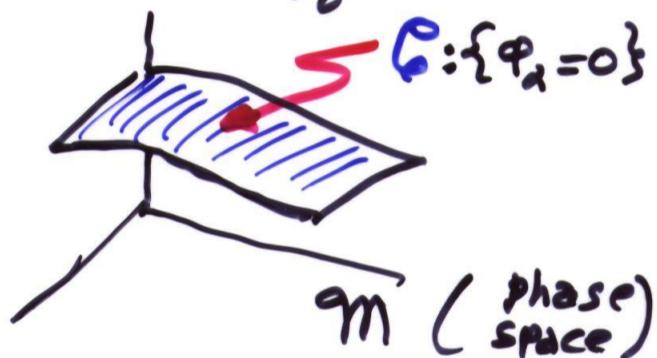
$$I = \int [p_j \dot{q}^j - H(p, q) - \lambda^\alpha \varphi_\alpha(p, q)] dt \quad \begin{matrix} 1 \leq j \leq J \\ 1 \leq \alpha \leq A \end{matrix}$$

e.g.m. $\ddot{q}^j = \frac{\partial H}{\partial p_j} + \lambda^\alpha \frac{\partial \varphi_\alpha}{\partial p_j} \Rightarrow \ddot{p}_j = -\frac{\partial H}{\partial q^j} - \lambda^\alpha \frac{\partial \varphi_\alpha}{\partial q^j}, \varphi_\alpha = 0$

$$\dot{q}^j = \{q^j, H\} + \lambda^\alpha \{q^j, \varphi_\alpha\}$$

$$\dot{p}_j = \{p_j, H\} + \lambda^\alpha \{p_j, \varphi_\alpha\}$$

$$\dot{\varphi}_\alpha = 0$$



Consistency requirement

$$\dot{\varphi}_\alpha(p, q) = \{\varphi_\alpha, H\} + \lambda^\beta \{\varphi_\alpha, \varphi_\beta\} = 0$$

There are two extreme cases:

First class start on C , stay on C freely

$$\{\varphi_\alpha, \varphi_\beta\} = c_{\alpha\beta}^\gamma \varphi_\gamma$$

$$\{\varphi_\alpha, H\} = h_\alpha^\beta \varphi_\beta$$

- a) $C_{\alpha\beta}^\gamma$ numbers: closed 1st class (YM)
 b) $C_{\alpha\beta}^\gamma$ functions: open 1st class (GR)

N.B. $\{\lambda^\alpha\}$ are undetermined by e.o.m.
 Must choose $\{\lambda^\alpha\}$ ("gauge") to solve e.o.m.

Second class start on \mathcal{G} , stay on \mathcal{G} by force

e.g., $\{\varphi_\alpha, \varphi_\beta\} \neq 0$ on \mathcal{G} , and has inverse on \mathcal{G}

$$\lambda^\beta = -\{\varphi_\alpha, \varphi_\beta\}^{-1}\{\varphi_\alpha, H\}$$

N.B. $\{\lambda^\alpha\}$ are determined by e.o.m.

- Also mixed 1st class & 2nd class systems

Quantization

1. Quantize before reduction
2. Reduce before quantization

1. Dirac

$$\varphi_\alpha(P, \delta) \rightarrow \bar{\varphi}_\alpha(P, Q)$$

$$\bar{\varphi}_\alpha |\psi_{\text{phys}}\rangle = 0, \quad |\psi_{\text{phys}}\rangle \in \mathcal{H}_{\text{phys}} \subset \mathcal{H}$$

$$\text{a)} [\bar{\varphi}_\alpha, \bar{\varphi}_\beta] |\psi_{\text{phys}}\rangle = 0 ?, \quad \text{b)} \langle \psi_{\text{phys}} | \psi_{\text{phys}} \rangle < \infty ?$$

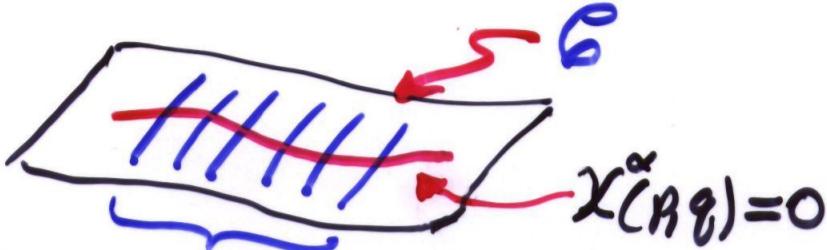
Works for some 1st class systems

2. Faddeev

$$q_m \int e^{i\frac{1}{\hbar} \int [P_j \dot{q}^j - H(P, \delta) - \lambda^\alpha \varphi_\alpha(P, \delta)] dt} dP d\delta d\lambda$$

$$= m' \int e^{i\frac{1}{\hbar} \int [p_j \dot{g}^j - H(p, g)] dt} \delta\{\varphi\} dp dg$$

Most likely diverges — choose a (dynamical) gauge



gauge equivalent sets of variables

$$\{x^\alpha, \varphi_\beta\} \equiv \Delta^\alpha_\beta$$

$$\det(\Delta^\alpha_\beta) \neq 0$$

$$m' \int e^{i\frac{1}{\hbar} \int [p_j \dot{g}^j - H(p, g)] dt} \delta\{x\} \det\{x, \varphi\} \delta\{\varphi\} dp dg$$

$$= m^* \int e^{i\frac{1}{\hbar} \int [p_B^* \dot{g}^{*B} - H^*(p^*, g^*)] dt} \delta p^* dg^*$$

- Do quantization and reduction commute?

Consider the following example:

$$I = \int [p \dot{g} - \lambda(p^2 + g^4 - E)] dt$$

question: What E values yield a quantization?

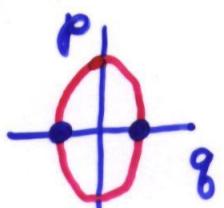
1. Dirac

$$(p^2 + q^4 - E) |\psi_{phys}\rangle = 0$$

Therefore, $E \in \{E_n\}$, set of eigenvalues, work

2. Faddeev

$$m \int e^{i\frac{1}{\hbar} \int p \dot{g} dt} \delta\{p^2 + g^4 - E\} dp dg$$



gauge: $\chi = -p = 0$, $\{x, \varphi\} = 4g^3$

$$I = m' \int e^{i\frac{1}{\hbar} \int p \dot{g} dt} \delta\{p\} (\pi_t 4g^3) \delta\{p^2 + g^4 - E\} dp dg$$

$$= m' \int (\pi_t 4g^3) \delta\{g^4 - E\} \pi_t dg$$

- i) $g > 0 : I = \text{const.} > 0$; ii) $g \geq 0 : I = 0$

Result: No restriction on E

before proceeding we introduce —

Another Property of Coherent States: Reproducing Kernel Hilbert Spaces

$\{|\ell\rangle\}$, set of continuous vectors that spans \mathcal{H}

Two elements in a dense set of vectors:

$$|\psi\rangle \equiv \sum_{j=1}^J \alpha_j |\ell_j\rangle , J < \infty$$

$$|\varphi\rangle \equiv \sum_{k=1}^K \beta_k |\ell_{(k)}\rangle , K < \infty$$

Functional representation:

$$\psi(\ell) \equiv \langle \ell | \psi \rangle = \sum_{j=1}^J \alpha_j \langle \ell | \ell_j \rangle$$

$$\varphi(\ell) \equiv \langle \ell | \varphi \rangle = \sum_{k=1}^K \beta_k \langle \ell | \ell_{(k)} \rangle$$

Inner product:

$$(\psi, \varphi) = \langle \psi | \varphi \rangle = \sum_{j,k=1}^{J,K} \alpha_j^* \beta_k \langle \ell_j | \ell_{(k)} \rangle$$

Complete space in norm $\|\psi\| = (\psi, \psi)^{1/2}$

N.B. All from Reproducing Kernel: $\langle \ell' | \ell \rangle$

Criteria for a kernel $\mathcal{K}(\ell'; \ell)$ to work:

1) continuity in ℓ' & ℓ

2) $\sum_{j,k=1}^{J,J} \alpha_j^* \alpha_k \mathcal{K}(\ell_j; \ell_k) \geq 0 , J < \infty$

N.B. Let $\mathcal{K}_c(\ell'; \ell) = c \mathcal{K}(\ell'; \ell)$, $c > 0$

$\psi_c(\ell) = \sum \alpha_j \mathcal{K}_c(\ell; \ell_j) = c \psi(\ell)$; $\varphi_c(\ell) = c \varphi(\ell)$

$(\psi_c, \varphi_c)_c = \sum \alpha_j^* \beta_k \mathcal{K}_c(\ell_j; \ell_{(k)}) = c (\psi, \varphi)$

Same space of functions; different inner prod.

Projection Operator Method

Quantize first: $\Phi_\alpha(P, q) \rightarrow \bar{\Phi}_\alpha(P, Q)$

focus on $\sum_x \bar{\Phi}_\alpha^2$ and note that

$$\bar{\Phi}_\alpha |\psi_{\text{phys}}\rangle = 0 \iff \sum_x \bar{\Phi}_\alpha^2 |\psi_{\text{phys}}\rangle = 0$$

Extend the Dirac procedure as follows:

Projection operator: $E = E^2 = E^\dagger$; choose

$$E = E(\sum_x \bar{\Phi}_\alpha^2 \leq \delta^2(\hbar)) \equiv \int_0^{\delta^2(\hbar)} dE (\sum_x \bar{\Phi}_\alpha^2)$$

$$\Psi_{\text{phys}} = E \Psi \quad (\text{regularized by } \delta^2)$$

Final step: Reduce δ appropriately!

Three basic examples:

① $E(J_1^2 + J_2^2 + J_3^2 \leq \hbar^2/2) \Rightarrow J_k = 0 \text{ all } k$

② $E(P^2 + Q^2 \leq \hbar) \Rightarrow (Q + iP) |\psi_{\text{phys}}\rangle = 0$

if P & Q irreducible, $E = |0\rangle\langle 0|$

Remark: Examples ① & ② have operators with discrete spectrum

① spectrum includes zero (1st class constraints)

② spectrum excludes zero (2nd class constraints)

③ $E(Q^2 \leq \delta^2)$, zero in continuous spectrum

Sketch of issues:

$$\langle \varphi | E(Q^2 \leq \delta^2) | \varphi \rangle = \int_{-\delta}^{\delta} |\varphi(x)|^2 dx \xrightarrow{\delta \rightarrow 0} 0$$

$$\frac{1}{2\delta} \langle \varphi | E(Q^2 \leq \delta^2) | \varphi \rangle = \frac{1}{2\delta} \int_{-\delta}^{\delta} |\varphi(x)|^2 dx \xrightarrow{\delta \rightarrow 0} |\varphi(0)|^2$$

provided $\varphi(x)$ continuous at zero

A more careful way to proceed:
use coherent states to take a
form limit

$$\mathcal{K}(p', q'; p, q) \equiv \frac{\langle p', q' | E(Q^2 \leq \delta^2) | p, q \rangle}{\langle 0, 0 | E(Q^2 \leq \delta^2) | 0, 0 \rangle}$$

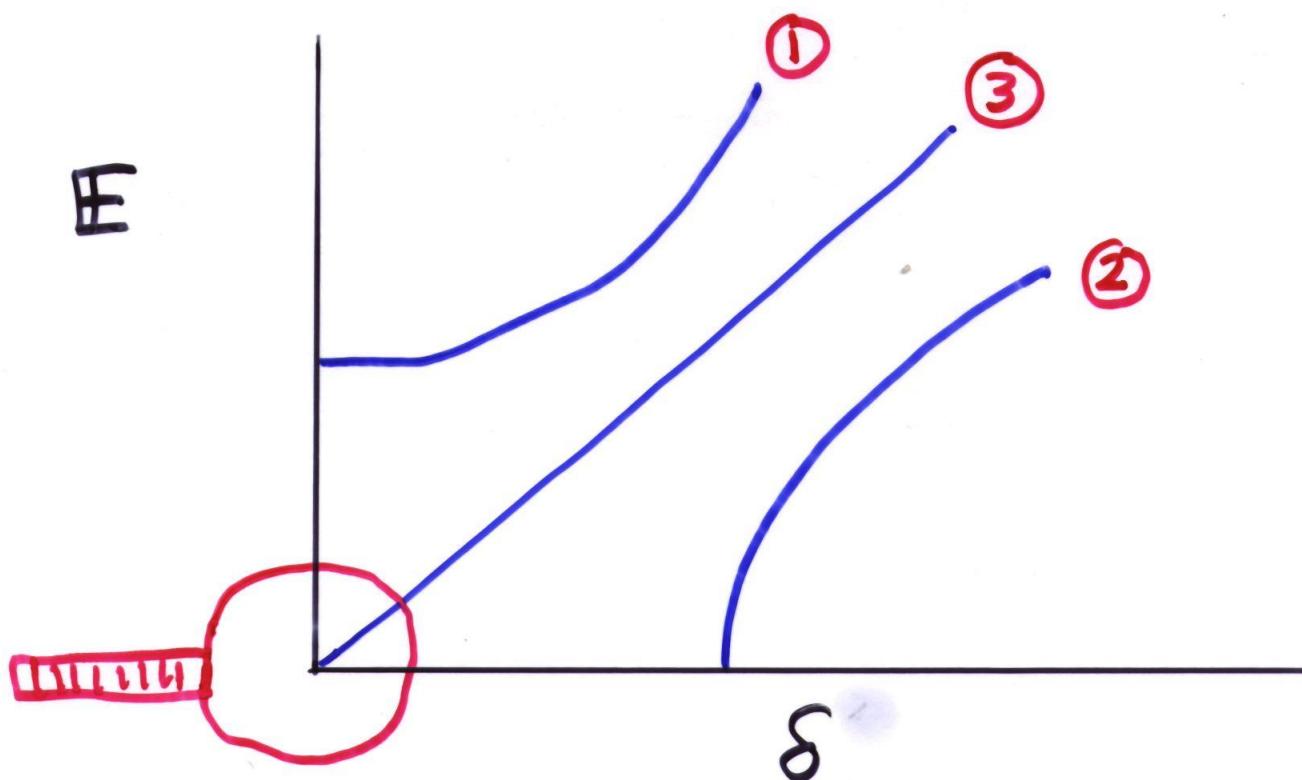
$$= \frac{\int_{-\delta}^{\delta} \langle p', q' | x \rangle \langle x | p, q \rangle dx}{\int_{-\delta}^{\delta} \langle 0, 0 | x \rangle \langle x | 0, 0 \rangle dx}$$

As $\delta \rightarrow 0$,

$$\mathcal{K}_0(p', q'; p, q) \equiv \frac{\langle p', q' | x=0 \rangle \langle x=0 | p, q \rangle}{\langle 0, 0 | x=0 \rangle \langle x=0 | 0, 0 \rangle}$$

Needs only a continuous fiducial vector $\eta(x)$.
Here, \mathcal{K}_0 is a RK for a one-dimensional
physical Hilbert space, $\mathcal{H}_{\text{phys}}$

Rough graphical summary



An operator identity: Integral representation for \mathbb{E}

$$\mathbb{E}(\sum_{\alpha} \bar{\Phi}_{\alpha}^2 \leq \delta^2(t_1)) = \int \mathcal{T} e^{-i \int_{t_1}^{t_2} \lambda(t) \bar{\Phi}_{\alpha} dt} dR(\lambda)$$

$R(\lambda)$ depends on A , $t_2 > t_1$, and δ^2 — it is completely independent of the set $\{\bar{\Phi}_{\alpha}\}_{\alpha=1}^A$

The integral representation can be used in path integral constructions such as

$$\langle p_i^u g^u | \mathbb{E} e^{-i(\mathbb{E} H \mathbb{E}) T/\hbar} \mathbb{E} | p_i' g' \rangle$$

$$= \mathcal{M} \int e^{i \frac{1}{\hbar} \int [p_i \dot{g}^i - H(p, g) - \lambda^{\alpha} \varphi_{\alpha}(p, g)] dt} Dp Dg dR(\lambda)$$

valid for 1st & 2nd class constraints

Comparison with Traditional Methods

Traditional Methods

- Gauge fixing
- Faddeev-Popov det.
- Gribov ambiguity
- Auxiliary variables
e.g., ghosts

Dirac brackets

:

Projection Operator Method

- { No gauge fixing
- { No auxiliary variables
- { No need to eliminate 2nd class constraints

:

Two advanced examples

④ Classical

$$I = \int [P\dot{q} - \lambda q^3(2-q)] dt$$

$$\dot{q}=0, \dot{p}=-3\lambda q^2(2-q) + \lambda q^3, q^3(2-q)=0$$

Solution of e.o.m.

$$q(t) = q(0), q(0) = 0, 2 \text{ since } q^3(2-q) = 0$$

$$\begin{aligned} p(t) &= p(0) - 3q^2(0)(2-q(0)) \int_0^t \lambda(t') dt' \\ &\quad + q^3(0) \int_0^t \lambda(t') dt' \end{aligned}$$

$$\therefore q(t) = 0, p(t) = p(0) \text{ when } q(0) = 0$$

$$q(t) = 2, p(t) = p(0) + 8 \int \lambda(t') dt' \text{ if } q(0) = 2$$

The 'physical', reduced classical phase space is

$$(P, q) = (P, 0) + (P, 2)$$

N.B. p for $(P, 0)$ is gauge independent
 p for $(P, 2)$ is gauge dependent

N.B. The constraint at $q=2$ is called a regular constraint; the constraint at $q=0$ is called an irregular constraint

Quantum

$$\begin{aligned} E(Q^6(2-Q)^2 \leq \delta^2) &= E(-\delta \leq Q^3(2-Q) \leq \delta) \\ &= E(-\delta \leq Q^3(2-Q) \leq \delta), 0 < \delta \ll 1 \\ &= E(-\delta \leq 2Q^3 \leq \delta) + E(-\delta \leq 8(2-Q) \leq \delta) \\ &\equiv E_0(-\delta_0 \leq Q \leq \delta_0) + E_2(-\delta_2 \leq Q \leq 2+\delta_2) \end{aligned}$$

where $\delta_0 \equiv (\delta/2)^{1/3}$, $\delta_2 \equiv \delta/8$

Use coherent state matrix elements to make Reproducing Kernels

$$\begin{aligned} \mathcal{K}(p', g'; p, g) &\equiv \langle p', g' | E_0 | p, g \rangle + \langle p', g' | E_2 | p, g \rangle \\ &\equiv \mathcal{K}_0(p', g'; p, g) + \mathcal{K}_2(p', g'; p, g) \end{aligned}$$

The fact that $E_0 E_2 = 0$ leads to

$$\begin{aligned} \int \mathcal{K}_0(p'', g''; p, g) \mathcal{K}_2(p, g; p', g') d\mu(p, g) \\ = \langle p'', g'' | E_0 E_2 | p', g' \rangle = 0 \end{aligned}$$

Introduce the function ($a > 0, b > 0$)

$$A(p', g'; p, g) \equiv \sqrt{a} \frac{\mathcal{K}_0(p', g'; p, g)}{\sqrt{\langle 0,0 | E_0 | 0,0 \rangle}} + \sqrt{b} \frac{\mathcal{K}_2(p', g'; p, g)}{\sqrt{\langle 0,2 | E_2 | 0,2 \rangle}}$$

and define the new Reproducing Kernel

$$\begin{aligned} \hat{\mathcal{K}}(p'', g''; p', g') &\equiv \int A(p'', g''; p, g) \mathcal{K}(p, g; p', g') A(\bar{p}, \bar{g}; p', g') \\ &\quad \times d\mu(p, g) d\mu(\bar{p}, \bar{g}) \\ &= a \frac{\langle p'', g'' | E_0 | p', g' \rangle}{\langle 0,0 | E_0 | 0,0 \rangle} + b \frac{\langle p'', g'' | E_2 | p', g' \rangle}{\langle 0,2 | E_2 | 0,2 \rangle} \end{aligned}$$

Now, take limit $\delta \rightarrow 0$ to yield another RK

$$\tilde{\mathcal{K}}(p'', g''; p', g') = a \frac{\langle p'', g'' | x=0 \rangle \langle x=0 | p', g' \rangle}{\langle 0,0 | x=0 \rangle \langle x=0 | 0,0 \rangle} + b \frac{\langle p'', g'' | x=2 \rangle \langle x=2 | p', g' \rangle}{\langle 0,2 | x=2 \rangle \langle x=2 | 0,2 \rangle}$$

which defines a two-dimensional $\mathcal{H}_{\text{phys}}$

Specialize to

$$|P, g\rangle \equiv e^{iP\theta/\hbar} e^{-i\theta P/\hbar} |0\rangle, \quad (Q+iP)|0\rangle = 0$$

which leads to

$$\tilde{\mathcal{K}}(p'', g''; p', g') = e^{-(g''^2 + g'^2)/2\hbar} \\ + e^{-i\alpha(p'' - p')/\hbar} e^{-[(g''-2)^2 + (g'-2)^2]/2\hbar}$$

N.B. In first term "Q=0", in second term "Q=2"

Observables in 'physical' Hilbert space

Observable Θ obeys $[\Theta, E] = 0$

Observable part of G , $G^E \equiv EGE$

Observable momentum & coordinate

$$P^E = EPE, \quad Q^E = EQE$$

Classical (" \hbar -augmented") momentum

$$P^P(p, g; \hbar) = \lim_{\delta \rightarrow 0} \frac{a \langle P, g | E_0 P E_0 | P, g \rangle + b \langle P, g | E_2 P E_2 | P, g \rangle}{a \langle P, g | E_0 | P, g \rangle + b \langle P, g | E_2 | P, g \rangle} = p$$

$$\therefore P^P(p, g) = \lim_{\hbar \rightarrow 0} P^P(p, g; \hbar) = p$$

meaning that the range of the variable $P^P(p, g)$ is the same as p , i.e., all of \mathbb{R}

Classical (" \hbar -augmented") coordinate

$$q^P(p, g; \hbar) = \lim_{\delta \rightarrow 0} \frac{a \langle P, g | E_0 Q E_0 | P, g \rangle + b \langle P, g | E_2 Q E_2 | P, g \rangle}{a \langle P, g | E_0 | P, g \rangle + b \langle P, g | E_2 | P, g \rangle}$$

$$g^P(p, \theta; \hbar) = \frac{2be^{-(\theta-2)^2/\hbar}}{ae^{-\theta^2/\hbar} + be^{-(\theta-2)^2/\hbar}}$$

To obtain true classical limit

$$g^P(p, \theta) = \lim_{\hbar \rightarrow 0} g^P(p, \theta; \hbar) = \begin{cases} 0, & \theta < 1 \\ 2, & \theta > 1 \end{cases}$$

and we recover the original 'physical' reduced phase space:

$$(p, \theta) = (p, 0) \oplus (p, 2), \quad p \in \mathbb{R}$$

(Observe that

$$g^P(p, 1) = \lim_{\hbar \rightarrow 0} g^P(p, 1; \hbar) = \frac{2b}{a+b}$$

holds only for a set of measure zero in the original classical phase space.)

⑤ An example with an "anomaly"

Classical: Three examples in parallel

a) $\Phi_k = j_k = \epsilon_{kem} p_e g_m = 0$

$$\{j_k, j_\ell\} = \epsilon_{kem} j_m \quad \text{1st class (closed)}$$

b) $\Phi_k = r_k \equiv f j_k = 0, \quad f = (1-\alpha) + \alpha (p_k^2 + g_k^2)/\hbar$

$$\{r_k, r_\ell\} = \epsilon_{kem} f(p, \theta) r_m \quad \text{1st class (open)}$$

c) $\Phi_k = s_k \equiv g j_k = 0, \quad g = (1 - 3\beta/2) + \beta \left[\frac{(p_k^2 + g_k^2)}{\hbar} + \frac{(p_\ell^2 + g_\ell^2)}{2\hbar} \right]$

$$\{s_k, s_\ell\} = \alpha_{kem}(p, g) s_m \quad \text{1st class (open)}$$

Quantum: Three examples in parallel

$$\vec{J}_k \rightarrow J_k = \epsilon_{kem} P_e Q_m = i\hbar a_e^\dagger \epsilon_{kem} a_m$$

$$N = a_k^\dagger a_k = N_1 + N_2 + N_3$$

$$f \rightarrow F = 1 + 2\alpha(1 + N) , [F, J_k] = 0$$

$$g \rightarrow G = 1 + \beta(2N_1 + N_2) , [G, J_k] \neq 0$$

$$N.B. [N, J_k] = [N, F] = [N, G] = 0$$

Restrict study to $N=2$ subspace

$$\left\{ |2,0,0\rangle, |0,2,0\rangle, |0,0,2\rangle \right\}$$

$$\left\{ |0,1,1\rangle, |1,0,1\rangle, |1,1,0\rangle \right\}$$

a) $E(\sum J_k^2 \leq \hbar^2) = |0_J\rangle \langle 0_J| , \sum J_k^2 |0_J\rangle = 0$

$$|0_J\rangle \propto |2,0,0\rangle + |0,2,0\rangle + |0,0,2\rangle$$

b) $E(\sum R_k^2 = F^2 \sum J_k^2 \leq \hbar^2) = |0_R\rangle \langle 0_R| , \sum R_k^2 |0_R\rangle = 0$

$$|0_R\rangle = |0_J\rangle \propto |2,0,0\rangle + |0,2,0\rangle + |0,0,2\rangle$$

Both $\sum J_k^2$ & $\sum R_k^2$ are 1st class quantum constraints because spectrum has zero

c) $S_k = (G J_k + J_k G)/2$

$$E(\sum S_k^2 \leq \hbar^2) = |0_S\rangle \langle 0_S|$$

$$\sum S_k^2 |0_S\rangle = (\hbar^2 \beta/2 + \dots) |0_S\rangle$$

for $0 < \beta \ll 1 \therefore$ 2nd class

$$|0_S\rangle \propto (1 - 2\beta + \dots) |2,0,0\rangle + (1 - \beta + \dots) |0,2,0\rangle + |0,0,2\rangle$$

4. QUANTUM GRAVITY

Electromagnetism, Classical

Gravity is a gauge theory somewhat like E&M

$$I = \int [A_{\mu,\nu} F^{\mu\nu} + \frac{1}{4} F_{\mu\nu} F^{\mu\nu}] d^4x$$

$$F_{\mu\nu} = 0, \quad A_{\nu\mu} - A_{\mu\nu} = F_{\mu\nu}$$

$$\text{let } E^i = F^{0i}, \quad B^k = \frac{1}{2} \epsilon^{ijk} F_{ij}$$

$$I = \int [-E^i \dot{A}_i - \frac{1}{2} (E^i E^i + B^i B^i) - A_0 E^i_{,i}] d^3x dt$$

Quantum ($\hbar = 1$)

$$\text{operators: } [E^i(x), A_k(y)] = i \delta^i_k \delta(x-y)$$

$$\text{c-numbers: } \xi = \xi_T + \xi_L, \quad \nabla \cdot \xi_T = 0 = \nabla \times \xi_L$$

$$\alpha = \alpha_T + \alpha_L, \quad \nabla \cdot \alpha_T = 0 = \nabla \times \alpha_L$$

$$|\xi, \alpha\rangle = e^{i \int [\alpha \cdot \xi - \xi \cdot \alpha] d^3x} |0\rangle$$

$$\langle \xi'', \alpha'' | \xi', \alpha' \rangle = e^{i \frac{1}{2} \sum_{j=L}^T \int [\xi''_j \cdot \alpha'_j - \alpha''_j \cdot \xi'_j] d^3x}$$

$$e^{-\frac{1}{4} \sum_{j=L}^T \left\{ \left| \frac{\xi''_j - \xi'_j}{\omega} \right|^2 + \omega |\tilde{\alpha}'_j - \tilde{\alpha}''_j|^2 \right\} d^3k}$$

$$\text{Use } b = \nabla \times \alpha \equiv \nabla \times \alpha_T \equiv b_T, \quad \omega^2 \tilde{\alpha}_T^2 = \tilde{b}_T^2, \quad \omega = |k_T|$$

Introduce dynamics plus constraint

$$\begin{aligned} \lim_{\delta \rightarrow 0} \frac{\langle \xi'', \alpha'' | e^{-iHt} E | \xi', \alpha' \rangle}{\langle 0, \alpha | E | 0, 0 \rangle} &= e^{i \frac{1}{2} \int [\xi''_T \cdot \alpha'_T(t) - \alpha''_T \cdot \xi'_T(t)] d^3x} \\ &\times e^{-\frac{1}{4} \int \left\{ \frac{[\xi''_T(x) - \xi'_T(x,t)] \cdot [\xi''_T(y) - \xi'_T(y,t)] + [b''_T(x) - b'_T(x,t)] \cdot [b''_T(y) - b'_T(y,t)]}{(x-y)^2} \right\}}, \end{aligned}$$

$d^3x d^3y$

$$\times e^{-\frac{1}{2} \int \left\{ \frac{\xi''_L(x) \cdot \xi''_L(y)}{(x-y)^2} + \frac{\xi'_L(x) \cdot \xi'_L(y)}{(x-y)^2} \right\} d^3x d^3y}$$

No.B. $\xi'_T(t)$, $\alpha'_T(t)$, $b'_T(t)$ evolve with classical dynamics

Classical gravity

$$I = \frac{c^4}{16\pi G} \int \sqrt{-g} R(g) d^4x \quad , \quad \frac{c^4}{16\pi G} = 1 = c = \hbar$$

First order form (Palatini)

$$I = \int \sqrt{-g} g^{\mu\nu} [\Gamma_{\mu\nu}^\alpha - \Gamma_{\mu\alpha}^\alpha + \Gamma_{\mu\nu}^\alpha \Gamma_{\alpha\rho}^\beta - \Gamma_{\mu\rho}^\alpha \Gamma_{\nu\alpha}^\beta] d^4x$$

$$\text{Vary } g^{\mu\nu} \text{ get } R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0$$

$$\text{Vary } \Gamma_{\mu\nu}^\alpha \text{ recover definition of } \Gamma_{\mu\nu}^\alpha(g)$$

Make 3+1 split of "space" plus "time" (ADM)

$$\mu, \nu = 0, 1, 2, 3 ; \quad i, j, k = 1, 2, 3$$

$${}^4g_{\mu\nu} = \left[\begin{array}{c|c} g_{ij} & N_i \\ \hline & N_j - (N^2 - N_i N^j) \end{array} \right], \quad g^{jk} g_{kj} = \delta^j_i \quad \{g_{ij}\} > 0$$

$$\pi^{ij} \equiv \sqrt{-g} ({}^4\Gamma_{pq}^0 - g_{pq} {}^4\Gamma_{rs}^0 g^{rs}) g^{ip} g^{js}$$

$$I = \int \sqrt{-{}^4g} {}^4R d^4x \doteq \int [-g_{ab} \dot{\pi}^{ab} - N^a H_a - NH] d^3x dt$$

$$\dot{g}_{ab} = \dots, \quad \dot{\pi}^{ab} = \dots,$$

$$H_a = -2 \pi_a^b \dot{g}_{ib} = 0$$

$$H = \frac{1}{\sqrt{g}} [\pi_a^a \pi_b^b - \frac{1}{2} \pi_a^a \pi_b^b] + \sqrt{g} R = 0$$

Fundamental Poisson brackets

$$\{g_{ab}(x), g_{cd}(y)\} = 0 = \{\pi^{ab}(x), \pi^{cd}(y)\}$$

$$\{g_{ab}(x), \pi^{cd}(y)\} = \frac{1}{2} (\delta_a^c \delta_b^d + \delta_a^d \delta_b^c) \delta(x, y)$$

Constraint brackets

$$\{H_a(x), H_b(y)\} = \delta_{,a}(x,y) H_b(x) - \delta_{,b}(x,y) H_a(x)$$

$$\{H_a(x), H(y)\} = \delta_{,a}(x,y) H(x)$$

$$\{H(x), H(y)\} = \delta_{,a}(x,y) g^{ab}(x) H_b(x)$$

Leads to open 1st class constraint system

$16 = (6+6+4)$ equations of motion for 16 variables (g_{ab}, π^{ab}, N^a, N)

However N^a, N are not determined by e.o.m.

They need to be chosen to find solution

Fixing N^a, N amounts to choosing future coordinate system

One possibility is to reduce to physical variables. So solve constraints for 4 variables (among g_{ab}, π^{ab}), then impose "coordinate conditions" (i.e., dynamical gauge fixing — the $\chi^\alpha = 0$ of Faddeev), to eliminate 4 more leaving $6+6-4-4=4 = "2p"+"2\theta"$ variables. A difficult nonlinear problem; besides the remaining $2p, 2\theta$ are unlikely to be Cartesian. A major "road block"!

Kinematics

Projection operator method insists on no reduction before quantization. One must quantize all $g_{ij} \rightarrow \hat{g}_{ij}$, $\pi^{ij} \rightarrow \hat{\pi}^{ij}$, subject to $\{\hat{g}_{ij}\} > 0$, and then reduce

Adopt standard commutation relations:

$$[\hat{g}_{ab}(x), \hat{\pi}^{cd}(y)] = \frac{i}{2}(\delta_a^c \delta_b^d + \delta_a^d \delta_b^c) \delta(x, y)$$

infinitely many unitarily inequivalent, irreducible representations

However, all representations imply that

$$\begin{aligned} & e^{i \int U_{ab} \hat{\pi}^{ab} dy} \hat{g}_{cd}(x) e^{-i \int U_{ab} \hat{\pi}^{ab} dy} \\ &= \hat{g}_{cd}(x) + U_{cd}(x) \equiv \hat{g}_{cd}^u : \text{no longer pos. def.} \end{aligned}$$

HELP from a single degree of freedom

Recall affine variables

$$[Q, P] = i \cdot, \quad Q > 0$$

$$e^{iuP} Q e^{-iuP} = Q + u : \text{no longer pos. def.}$$

$$\text{solution } [Q, P] Q = [Q, D] = iQ, \quad Q > 0$$

$$D = \frac{1}{2}(QP + PQ), \quad e^{iuD} Q e^{-iuD} = e^u Q > 0$$



Suggests replacing $\pi^{ab}(x)$ by

$$\pi_c^a(x) \equiv \pi^{ab}(x) g_{bc}(x)$$

i.e., "lower" one index on momentum

$\pi_c^a(x)$ is called momentric (momentum + metric)

Leads to new classical Poisson brackets:

$$\{\pi_b^a(x), \pi_d^c(y)\} = \frac{1}{2} [\delta_b^c \pi_d^a(x) - \delta_d^a \pi_b^c(x)] \delta(x, y)$$

$$\{g_{ab}(x), \pi_d^c(y)\} = \frac{1}{2} [\delta_a^c g_{bd}(x) + \delta_b^c g_{ad}(x)] \delta(x, y)$$

$$\{g_{ab}(x), g_{cd}(y)\} = 0, \quad \{g_{ab}(x)\} > 0$$

New classical variables are as good as original variables because

$$\pi^{ab}(x) = g^{ac}(x) \pi^c_a(x)$$

New Poisson brackets close to 'Lie' algebra

Let us quantize new variables: $\hat{g}_{ab}, \hat{\pi}_a^c$

Quantum commutators

$$[\hat{\pi}_b^a(x), \hat{\pi}_d^c(y)] = \frac{i}{2} [\delta_b^c \hat{\pi}_d^a(x) - \delta_d^a \hat{\pi}_b^c(x)] \delta(x,y)$$

$$[\hat{g}_{ab}(x), \hat{\pi}_d^c(y)] = \frac{i}{2} [\delta_a^c \hat{g}_{bd}(x) + \delta_b^c \hat{g}_{ad}(x)] \delta(x,y)$$

$$[\hat{g}_{ab}(x), \hat{g}_{cd}(y)] = 0, \quad \{ \hat{g}_{ab}(x) \} > 0$$

Direct algebraic consequence

$$e^{i \int y_b^a \hat{\pi}_a^b d^3y} \hat{g}_{cd}(x) e^{-i \int y_b^a \hat{\pi}_a^b d^3y}$$

$$= \{ e^{y(x)/2} \}_c^e \hat{g}_{ef}(x) \{ e^{y(x)/2} \}_d^f$$



which maintains positive definite property

and leads to self-adjoint representations

$$N.B. \hat{\pi}^{ab}(x) = " \frac{1}{2} [\hat{g}^{bc}(x) \hat{\pi}_c^a(x) + \hat{\pi}_c^a(x) \hat{g}^{bc}(x)] "$$

is not an operator (even after smearing)

Affine group algebra derives from

$$U[\pi, y] \equiv e^{i \int \pi^{ab}(y) \hat{g}_{ab}(y) d^3y} e^{-i \int y_b^a(y) \hat{\pi}_a^b(y) d^3y}$$

and linear combinations, of smooth functions

π^{ab} & y_b^a of, e.g., compact support.

Choose a representation

There exist infinitely many unitarily inequivalent, irreducible representations.

Equivalent to choosing $|n\rangle$ and defining

$$|\pi, \gamma\rangle \equiv e^{i\int \pi^{ab}(y) \hat{g}_{ab}(y) d^3y} e^{-i\int \gamma_b^a(y) \hat{\pi}_a^b(y) d^3y} |n\rangle$$

Basic properties of $|n\rangle$

$\langle n | \hat{g}_{ab}(x) | n \rangle \equiv \tilde{g}_{ab}(x)$ fixes topology and asymptotic nature of spacelike surface

$$\langle n | \hat{\pi}_b^a(x) | n \rangle = 0$$

Algebraic consequences

$$\langle \pi, \gamma | \hat{g}_{ab}(x) | \pi, \gamma \rangle = \{e^{\gamma(x)/2}\}_a^c \tilde{g}_{cd}(x) \{e^{\gamma(x)/2}\}_b^d \equiv g_{ab}(x)$$

$$\langle \pi, \gamma | \hat{\pi}_c^a(x) | \pi, \gamma \rangle = \pi^{ab}(x) g_{bc}(x) \equiv \pi_c^a(x)$$

HELP from a single degree of freedom

$$|p, g\rangle = e^{ipQ} e^{-i\ln(g)D} |n\rangle, \quad Q > 0, \quad g > 0$$

$$\langle n | Q | n \rangle = 1, \quad \langle n | D | n \rangle = 0$$

$$\langle p, g | Q | p, g \rangle = g, \quad \langle p, g | D | p, g \rangle = pg$$

$$\langle p'', g'' | p', g' \rangle = \left\{ \frac{g''^{-1/2} g'^{1/2}}{\left[\frac{1}{2}(g''^{-1} + g'^{-1}) + \frac{i}{2\beta}(p'' - p') \right]} \right\}^{2P}$$

Save for numerator, expression is an analytic function of $g'' + (i/\beta)p''$

Full coherent state overlap function

$$\langle \pi''\gamma'' | \pi'\gamma' \rangle =$$

$$\exp \left[-2 \int b(x) dx \ln \left(\frac{\det \left\{ \frac{1}{2} [g''^{ab}(x) + g'^{ab}(x)] + \frac{i}{2b(x)} [\pi''^{ab}(x) - \pi'^{ab}(x)] \right\}}{\det \{ g''^{ab}(x) \}^{1/2} \det \{ g'^{ab}(x) \}^{1/2}} \right) \right]$$

$$\equiv \langle \pi''\gamma'' | \pi'\gamma' \rangle$$

- 1) • Depends on $0 < b(x) < \infty$, a scalar density with dimensions L^{-3}
 - Different $b(x)$ correspond to inequivalent field operator representations
 - Expect $b(x)$ will disappear when constraints are fully enforced and final operator representation is attained
- 2) Only depends on $g_{ab}(x)$ and not on $\gamma_b^a(x)$
- 3) Invariant under spatial coordinate transformations

Functional integral representation

HELP from a single degree of freedom

Every functional representative

$$\psi(p, g) = \langle p, g | \psi \rangle$$

is analytic in $g + (i/\beta)p$ up to a factor:

$$B\psi(p, g) = [-ig^{-1}\partial_p + 1 + \beta^{-1}g\partial_g] \psi(p, g) = 0$$

(called a "polarization")

H
E
L
P

Let $A \equiv \frac{1}{2}\beta B^T B \geq 0$, then $A\psi(p, q) = 0$

$$\lim_{\nu \rightarrow \infty} (e^{-\nu TA}) S(p-p')S(q-q') = \Pi(p, q; p', q')$$

a projection kernel on zero subspace of A

But $\langle p, q | p', q' \rangle$ is proportional to this same kernel, therefore $\Pi \propto \langle p, q | p', q' \rangle$

Invoke Feynman-Kac-Stratonovich representation to learn that

$$\langle p'', q'' | p', q' \rangle = \lim_{\nu \rightarrow \infty} \mathcal{M}_\nu \int e^{-i \int g \dot{p} dt - \frac{i}{2\nu} \int [\rho^{-1} g^2 \dot{p}^2 + \rho g^2 \dot{g}^2] dt} d\rho dg$$

$$= \lim_{\nu \rightarrow \infty} 2\pi [1 - \frac{1}{2\rho}] e^{\nu T_2} \int e^{-i \int g dp} dW^\nu(p, q)$$

Gravity case

$$\langle \pi'', g'' | \pi', q' \rangle$$

$$= \lim_{\nu \rightarrow \infty} \mathcal{M}_\nu \int e^{-i/\nu \int g_{ab} \dot{\pi}^{ab} d^3x dt}$$

$$\times e^{-\frac{1}{2}\nu h \int [b(x)^{-1} g_{ab} \dot{g}_{cd} \dot{\pi}^{ab} \dot{\pi}^{cd} + b(x) g^{bc} g^{ad} \dot{g}_{ab} \dot{g}_{cd}] d^3x dt}$$

$$\times \left[\prod_{x,t} \prod_{a,b} d\pi^{ab}(x,t) dg_{ab}(x,t) \right]$$

Integration domain limited to $\{g_{ab}(x,t)\} > 0$

Note similarities with single degree of freedom case!

Quantum constraints

$H_a(x) \rightarrow \mathcal{H}_a(x)$ diffeomorphism constraints

$H(x) \rightarrow \mathcal{H}(x)$ temporal constraint

Ideally (Dirac):

$$\mathcal{H}_a(x)|\Psi_{\text{phys}}\rangle = 0, \quad \mathcal{H}(x)|\Psi_{\text{phys}}\rangle = 0$$

Constraint commutators:

$$[\mathcal{H}_a(x), \mathcal{H}_b(y)] = i\hbar [\delta_{ab}(x,y) \mathcal{H}_b(x) - \delta_{ba}(x,y) \mathcal{H}_a(x)]$$

$$[\mathcal{H}_a(x), \mathcal{H}(y)] = i\hbar \delta_{a0}(x,y) \mathcal{H}(x)$$

$$[\mathcal{H}(x), \mathcal{H}(y)] = \frac{i\hbar}{2} \delta_{ab}(x,y) [\hat{g}^{ab}(x) \mathcal{H}_b(x) + \mathcal{H}_b(x) \hat{g}^{ab}(x)]$$

The fact that (generally)

$$\hat{g}^{ab}(x) |\Psi_{\text{phys}}\rangle \notin \mathcal{H}_{\text{phys}}$$

means that the constraints have become partially second class (c.f., $Q|\Psi_p\rangle = 0, P|\Psi_p\rangle = 0$,

thus $[Q, P]|\Psi_p\rangle = i\hbar|\Psi_p\rangle = 0$, i.e. $\therefore |\Psi_p\rangle = 0$)

This is called an "anomaly". At this point most workers "change the theory"!

Projection operator method has no fear of second class constraints, and accepts the quantum constraints as given.

Introduce complete set of "orthonormal" functions $\{h_n(x)\}_{n=0}^{\infty}$ such that

$$\int h_n(x) h_m(x) b(x) d^3x = \delta_{nm}$$

$$b(x) \sum_{n=0}^{\infty} h_n(x) h_n(y) = \delta(x, y)$$

$$H_a(x) \rightarrow H_{a(n)} \equiv \int h_n(x) H_a(x) d^3x$$

$$H(x) \rightarrow H_{(n)} \equiv \int h_n(x) H(x) d^3x$$

$$\left\langle \sum_a \Phi_a^2 \right\rangle = \sum_{n=0}^{\infty} 2^{-n} \left\{ H_{(n)}^2 + \sum_a H_{a(n)}^2 \right\}$$

Can arrange a suitable $R(N^a, N)$ so that

$$\mathbb{E} \left(\sum_{n=0}^{\infty} 2^{-n} \left\{ H_{(n)}^2 + \sum_a H_{a(n)}^2 \right\} \leq \delta^2(t) \right)$$

$$= \int \mathcal{T} e^{-i \int [N_a^a(x,t) H_a(x) + N(x,t) H(x)] d^3x dt} \mathcal{D}R(N^a, N)$$

Insert into previous functional integral

$$\langle \pi^a g^a | \mathbb{E} | \pi^c g^c \rangle$$

$$\begin{aligned} &= \lim_{N \rightarrow \infty} \mathcal{M}_N \int e^{-i \int [g_{ab} \dot{\pi}^{ab} + N^a H_a^{(s)} + N H^{(s)}] d^3x dt} \\ &\quad \times e^{-i \int [b(x)^a g_{bc} g_{da} \dot{\pi}^{ab} \dot{\pi}^{cd} + b(x) g^{bc} g^{da} \dot{g}_{ab} \dot{g}_{cd}] d^3x dt} \\ &\quad \times \left[\prod_{x,t} \prod_{a,b} d\pi^{ab}(x,t) dg_{ab}(x,t) \right] \mathcal{D}R(N^a, N) \end{aligned}$$

N.B., $H_a^{(s)}(x)$ & $H^{(s)}(x)$ are symbols for $H_a(x)$ & $H(x)$, i.e., "t-augmented" functions

Ideally,

$$\lim_{\hbar \rightarrow 0} H_a^{(S)}(x) = H_a(x)$$

$$\lim_{\hbar \rightarrow 0} H^{(S)}(x) = H(x)$$

However, need to know what $H_a(x)$ and $H(x)$ are — including \hbar -dependent “counterterms” — to determine $H_a^{(G)}(x)$ and $H^{(W)}(x)$.

Counterterms for quantum gravity

Adopt the hard-core picture of nonrenormalizability

- Focus on scalar models

$$S_0(\hbar) = \eta_0 \int e^{\int \hbar \varphi d^3x - W_0(\varphi)} \partial \varphi$$

$$S_\lambda(\hbar) = \eta_\lambda \int e^{\int \hbar \varphi d^3x - W_0(\varphi) - \lambda V(\varphi)} \partial \varphi, \lambda > 0$$

Introduce

$$X(\varphi) \equiv \begin{cases} 1, & W_0(\varphi) < \infty, V(\varphi) < \infty \\ 0, & W_0(\varphi) < \infty, V(\varphi) = \infty \end{cases}$$

$$S_\lambda(\hbar) = \eta_\lambda \int e^{\int \hbar \varphi d^3x - W_0(\varphi) - \lambda V(\varphi)} X(\varphi) \partial \varphi, \lambda > 0$$

Renormalizable

$$X(\varphi) \equiv 1$$

$$S_\lambda(\hbar) \xrightarrow[\lambda \rightarrow 0]{} S_0(\hbar)$$

Nonrenormalizable

$$X(\varphi) \neq 1$$

$$S_\lambda(\hbar) \xrightarrow[\lambda \rightarrow 0]{} S'_0(\hbar) \neq S_0(\hbar)$$

Can argue that $X(\text{grav}) \neq 1$ as well!

What is known about \mathcal{G}_5^4 ?

$$\eta \int e^{S_0 \varphi d^5x} - \frac{1}{2} \int [(\nabla \varphi)^2 + m^2 \varphi^2] d^5x - \lambda \int \varphi^4 d^5x$$

- perturbation theory



$$\propto \frac{p^{2.5}}{p^{2.5}} \rightarrow \ln(1) \text{ needs } \varphi^6 \text{ term}$$



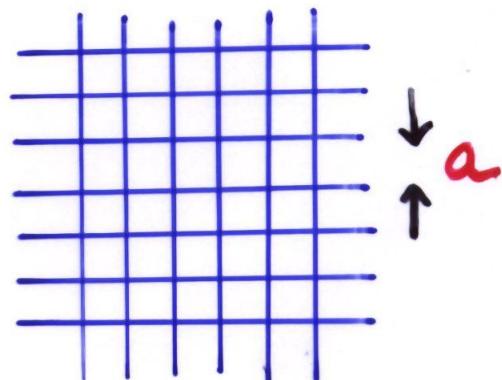
$$\propto \frac{p^{2.5}}{p^{2.4}} \rightarrow \lambda^2 \text{ needs } \varphi^8 \text{ term}$$

etc.; an infinite number of counterterms

- lattice regularization

$$N \iint e^{\sum h_k \varphi_k a^5 - \frac{1}{2} \sum (\varphi_{k+r} - \varphi_k)^2 a^3 - \frac{1}{4} m_a^2 a} \sum \varphi_k^4 a^5 \prod d\varphi_k$$

$$\xrightarrow[a \rightarrow 0]{} e^{\frac{1}{2} \int \varphi(x) U(x-y) \varphi(y) d^5x d^5y}$$



Hard core picture? YES

$$\int [(\nabla \varphi)^2 + m^2 \varphi^2] d^5x < \infty$$

$$\int \varphi^4 d^5x = \infty$$

$$\text{e.g., whenever } \varphi = \frac{e^{-|x|^p}}{|x|^p}, \quad 1.25 < p < 1.5$$

- Expansion of hard core leads to series of increasingly divergent contributions
- Lattice form lacks contribution from $\chi(\varphi)$