Problems for : Selected Topics in Quantum Theory

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Abstract

A set of problems is offered to accompany the set of lectures with the above title.

Topic 1: Coherent States

1.1 Let $\{|n\rangle\}$, $n = 0, 1, 2, 3, \ldots$, denote a complete set of orthonormal vectors in an abstract Hilbert space $\mathfrak{H}$, with $\langle m|n\rangle = \delta_{mn}$. Next, let

$$|\chi\rangle = \sqrt{1 - |\chi|^2}\,|0\rangle + \chi\,|1\rangle\ ,$$

for all complex χ, $0 \le |\chi| \le 1$, denote a family of *two*-dimensional vectors.

a) If $\chi \to \chi'$ in the sense of complex numbers, show that $|\chi\rangle \to |\chi'\rangle$ as vectors, namely that

$$\|\,|\chi\rangle - |\chi'\rangle\,\| \to 0\ ,$$

where $\|\,|\psi\rangle\,\| := \sqrt{\langle\psi|\psi\rangle}$. This exercise shows continuity of the labeling of the vectors.

b) Find a weight function $\rho(|\chi|) \ge 0$ such that

$$\int_{0\le|\chi|\le 1} |\chi\rangle\langle\chi|\ \rho(|\chi|)\ d\chi_r\,d\chi_i = |0\rangle\langle 0| + |1\rangle\langle 1| = I_2\ ,$$

where $\chi = \chi_r + i\chi_i$, and I_2 is the two-dimensional identity. This problem is an example of how a resolution of unity may be given as a positive (and continuous) superposition of one-dimensional projection operators.

With properties a) and b) established, i.e., continuity of labeling and the existence of a continuous resolution of unity, then one can assert that the set of states $\{|\chi\rangle\}$ constitutes a *set of coherent states.*

c) Let N be an operator such that $N|n\rangle = n|n\rangle$ for all n. Evaluate

$$\langle\chi|N|\chi\rangle \ .$$

It is useful sometimes to set $N = a^\dagger a$, where $a^\dagger$ and a denote creation and annihilation operators, respectively. In particular, we may choose these operators to have the commutation relation given by

$$[a, a^\dagger] = aa^\dagger - a^\dagger a = 1 \ .$$

Note any algebraic pattern in $N = a^\dagger a$ that also appears in $\langle\chi|N|\chi\rangle$.

This similarity between q-numbers and c-numbers lends itself to the interpretation of the diagonal coherent state matrix elements of an operator as being closely related to the classical variable. This association is an elementary example of the so-called *weak correspondence principle* that identifies the diagonal coherent state matrix elements of a quantum generator with the classical version of the same generator. Of course, this association leads to an $\hbar$-augmented classical expression since we have not as yet taken the limit in which $\hbar \to 0$.

d) With $\{b_{mn}\}_{m,n=1}^{2}$ a set of 4 complex constants, let

$$B = b_{00}\,|0\rangle\langle 0| + b_{01}\,|0\rangle\langle 1| + b_{10}\,|1\rangle\langle 0| + b_{11}\,|1\rangle\langle 1|$$

denote the most general 2×2 operator (matrix). Determine

$$B(\chi, \chi') \equiv \langle\chi|B|\chi'\rangle \ ,$$

and show that given only the *diagonal* matrix elements $B(\chi) \equiv B(\chi, \chi)$, the value of the *general* matrix elements $B(\chi, \chi')$ can be determined.

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This example illustrates how a limited set of coherent state expectation values of an operator determines the general operator matrix elements, i.e., determines the operator itself *uniquely*. This favorable property holds for a large variety of coherent states, but not for all sets of coherent states.

e) Consider the set of operators of the form, $|m\rangle\langle n|$, for all m, n, $0 \le m, n \le 1$. Find expressions $c_{m,n}(\chi)$ such that

$$|m\rangle\langle n| = \int c_{m,n}(\chi)\,|\chi\rangle\langle\chi|\,d\chi_r\,d\chi_i\ .$$

This example illustrates the "diagonal" representation of operators, namely, the representation of any operator as a superposition over coherent state projection operators.

f) Determine whether or not the set $\{c_{m,n}(\chi)\}$ is unique. Argue that any nonuniqueness arises because the present example is limited to a finite dimensional Hilbert space.

1.2 Consider the *three*-dimensional vectors

$$|\psi\rangle = M(|\psi|, b)\,[\,|0\rangle + \psi\,|1\rangle + b\psi^2\,|2\rangle]\ ,$$

for all $\psi \in \mathbb{C}$, where b is a fixed, real parameter, and $M(|\psi|, b)$ denotes a normalization factor such that $\|\,|\psi\rangle\,\|^2 \equiv \langle\psi|\psi\rangle = 1$ for all $|\psi\rangle$.

a) Find the range of allowed b values and an associated weight function $\sigma(|\psi|, b) \ge 0$ such that

$$\int |\psi\rangle\langle\psi|\,\sigma(|\psi|, b)\,d\psi_r\,d\psi_i = |0\rangle\langle 0| + |1\rangle\langle 1| + |2\rangle\langle 2| = I_3\ ,$$

the three-dimensional identity.

b) As an optional extension of this problem, one may add analogues of the additional questions from Exercise **1.1** for this set of three-dimensional coherent states. Specification of which extension is left to the instructor.

1.3 The *canonical coherent states* are defined by

$$|z\rangle \equiv e^{-|z|^2/2} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}}\,|n\rangle\ .$$

For this canonical set of coherent states, verify the several properties suggested in Exercise **1.1**, namely, answer analogues of all questions a) through f).

1.4 Consider the set of canonical coherent states introduced in Exercise **1.3** along with N, defined previously by the relation $N|n\rangle = n|n\rangle$, for all n.

a) Show that

$$e^{-iNt}|z\rangle = |e^{-it}z\rangle \ .$$

If the operator N denotes the Hamiltonian (in suitable units), then it follows that this example shows that

$$e^{-iNt}|z\rangle = |z(t)\rangle \ ,$$

namely that the time evolution of any coherent state is again a coherent state, a property which is referred to as *temporal stability.* Observe, that the more general statement

$$e^{-iNt}|z\rangle = |z,t\rangle$$

holds for any operator N and set of coherent states; temporal stability $|z,t\rangle = |z(t)\rangle$ is a significant restriction that applies only in special cases.

b) Alternative sets of coherent states may be defined by

$$|z,r\rangle = M(|z|,r)\sum_{n=0}^{\infty}\frac{z^n}{\sqrt{\rho_n}}|n\rangle \ ,$$

where $M(|z|,r)$ denotes a normalization factor; in particular,

$$M(|z|,r)^{-2} = \sum_{n=0}^{\infty}\frac{|z|^{2n}}{\rho_n}$$

for all $u = |z|^2 < U = \liminf_{n\to\infty}\rho_n$, where U, $U \le \infty$, denotes the radius of convergence for the series. Finally, we require that

$$\rho_n = \int_0^U u^n r(u)\,du$$

for some weight function $r(u) \geq 0$, normalized so that $\rho_0 = 1$. With ρ_n defined as moments of a normalized distribution, show that

$$\rho_1 \leq \rho_2^{1/2} \leq \rho_3^{1/3} \leq \cdots \rho_n^{1/n} \leq \cdots .$$

c) As a consequence, show that

$$\inf \lim_{n\to\infty} \rho_n = \lim_{n\to\infty} \rho_n .$$

d) In terms of the quantities defined above, find a weight function $K(|z|, r) \geq 0$ such that

$$\int |z,r\rangle\langle z,r|\, K(|z|,r)\, dx\, dy = \sum_{n=0}^{\infty} |n\rangle\langle n| = I_\infty \equiv I ,$$

where $z = x + iy$. In particular, show that if $r(u) = e^{-u}$, then the usual canonical coherent states emerge.

e) For the general coherent states $|z,r\rangle$ defined above, determine the form of the

i) the symplectic potential

$$d\Omega := i\,\langle z,r|\,d\,|z,r\rangle$$

ii) the induced metric

$$d\sigma^2 := 2\,[\,\|\,d\,|z,r\rangle\,\|^2 - |\,\langle z,r|\,d\,|z,r\rangle\,|^2]\ ,$$

and

iii) the scalar curvature associated with the given metric, all expressed as functions of the variables introduced above,.

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This exercise shows that the set of coherent states $\{\,|z,r\rangle\}$ induces a *metric* $d\sigma^2$ — and thereby a Riemannian geometry — on the set $\mathbb{C}$ of complex numbers.

1.5 A bounded operator B is called positive, i.e., $0 < B$, if $0 < \langle\psi|B|\psi\rangle$ for all nonzero $|\psi\rangle \in \mathfrak{H}$. For two positive operators B_1 and B_2, we write $B_1 < B_2$ provided $B_2 - B_1$ is a positive operator. Let A_n be an increasing sequence of positive operators such that $0 < A_1 < \cdots < A_n \cdots < I$. Assume that the $\lim_{n\to\infty}\langle\phi|\,(A_n - I)\,|\psi\rangle = 0$ holds for all $|\psi\rangle, |\phi\rangle \in \mathfrak{H}$ (called *weak* convergence). In that case, show that

$$\lim_{n\to\infty} \|\,(A_n - I)|\psi\rangle\| = 0$$

holds for all $|\psi\rangle \in \mathfrak{H}$ (called *strong* convergence).

The lesson of this example applies to the resolution of unity that coherent states enjoy. Let $\{|l\rangle\}$, $l \in \mathcal{L}$, a label space, denote a set of coherent states, and let δl denote a positively weighted absolutely continuous volume element on $\mathcal{L}$ such that

$$\int |l\rangle\,\langle l|\,\delta l = I\ ,$$

the identity on the (separable) Hilbert space in question. The question arises what kind of convergence is meant by this integral. Such an integral is called weakly convergent provided that for all $|\phi\rangle$ and $|\psi\rangle$ in $\mathfrak{H}$, it follows that

$$\int \langle\phi|l\rangle\,\langle l|\psi\rangle\,\delta l = \langle\phi|\psi\rangle\ .$$

In fact it is sufficient to show this relation holds when $|\phi\rangle = |\psi\rangle$, i.e., in the case that

$$\int |\langle l|\psi\rangle|^2\,\delta l = \langle\psi|\psi\rangle = \|\,|\psi\rangle\,\|^2\ .$$

This is the essential meaning for the resolution of identity, so one often sees the (correct) statement that the integral defining the resolution of unity converges weakly. However, more can be said. Consider the set of integrals given by

$$A_n := \int_{R_n} |l\rangle\,\langle l|\,\delta l\ ,$$

where R_n denotes a set of increasing ($R_n \subset R_{n+1}$) domains of integration that limit as $n \to \infty$ to the whole set $\mathcal{L}$. It is clear in this case that such operators obey the condition stated above, namely

$$0 < A_1 \cdots < A_n < \cdots < I \; .$$

Hence, we can conclude that the integral defining the resolution of unity converges strongly.

1.6 In an abstract sense, there is only one (separable) Hilbert space; however, there is a limitless number of different representations of this single abstract Hilbert space. Several examples will illustrate the point. The Hilbert space composed of square integrable functions on the real line, $\psi(x) \in L^2(\mathbb{R})$, with inner product determined by

$$(\psi, \phi) = \int \psi^*(x)\,\phi(x)\,dx$$

is the most commonly used representation in Schrödinger wave mechanics. However, this representation is unitarily equivalent to all other representations such as one based on a sequence space. In particular, let $\{h_n(x)\}_{n=0}^{\infty}$ denote a set of orthonormal functions (e.g., the Hermite functions) which enjoy the relations

$$\sum_{n=0}^{\infty} h_n(x)\,h_n(y) = \delta(x-y) \; ,$$

$$\int h_m(x)\,h_n(x)\,dx = \delta_{mn}$$

characteristic of an orthonormal sequence of functions. This set of functions leads to an alternative representation since we can introduce

$$a_n = \int h_n(x)\,\psi(x)\,dx \; .$$

$$\psi(x) = \sum_{n=0}^{\infty} a_n\,h_n(x) \; ,$$

along with the corresponding inner product

$$\int \psi^*(x)\,\phi(x)\,dx = \sum_{n=0}^{\infty} a_n^*\,b_n \; ,$$

showing the equivalence of the function $\psi(x)$ for all $x \in \mathbb{R}$ and the sequence $\{a_n\}_{n=0}^{\infty}$. In fact, the elements of $L^2(\mathbb{R})$ are strictly speaking *not* functions but equivalence classes composed of sets of functions that differ from one another on sets of measure zero. This mathematical nicety is often overlooked in physical studies of quantum systems because the typical form of a Hamiltonian forces the wave functions it acts on to be continuous, and as such we can, for most purposes, ignore its partner elements that differ on sets of measure zero.

This prelude is intended to introduce another, less familiar, class of Hilbert space representations, the so-called *reproducing kernel Hilbert spaces*, and the present exercise is intended to serve as an introduction to such spaces. Let an abstract Hilbert space $\mathfrak{H}$ be given and denote the vectors in this space by $|\psi\rangle \in \mathfrak{H}$, etc. Furthermore. choose a set of vectors $\{|l\rangle\}$ for all $l \in \mathcal{L}$, a topological label space locally isomorphic to $\mathbb{R}^p$ for some p. We require that the vectors $|l\rangle$ span the Hilbert space, and that they are continuously labeled, meaning that as $l \to l'$ in the topology of $\mathcal{L}$, that the associated vectors are strongly continuous, i.e.,

$$\lim_{l \to l'} \| \, |l\rangle - |l'\rangle \, \| = 0 \, .$$

It is often useful if the set $\{|l\rangle\}$ forms a set of coherent states, but this is not required.

We focus on a dense set of Hilbert space vectors formed by finite linear combinations of the set $\{|l\rangle\}$. Two typical vectors in this dense set are given by

$$|\psi\rangle = \sum_{n=1}^{N} a_n \, |l_n\rangle \, , \qquad N < \infty \, ,$$
$$|\phi\rangle = \sum_{m=1}^{M} b_m \, |l'_m\rangle \, , \qquad M < \infty \, .$$

The overlap function $\langle l|l'\rangle$ plays a central role in our discussion. As *functional representatives* of the abstract vectors we choose the continuous functions

$$\psi(l) \equiv \sum_{n=1}^{\infty} a_n \, \langle l|l_n\rangle = \langle l|\psi\rangle \, ,$$

$$\phi(l) \equiv \sum_{m=1}^{\infty} b_m \, \langle l | l'_m \rangle = \langle l | \phi \rangle \ ,$$

and as the inner product of these two elements we adopt the rule

$$(\psi, \phi) \equiv \sum_{n,m=1}^{N,M} a_n^* b_m \, \langle l_n | l'_m \rangle = \langle \psi | \phi \rangle \ .$$

Completion of the space leads to a reproducing kernel Hilbert space all of whose elements are *continuous functions*; every ingredient of this space is determined by the *reproducing kernel* $\langle l | l' \rangle$ which is a continuous function on $\mathcal{L} \times \mathcal{L}$.

If, in addition, the set of states $\{|l\rangle\}$ form a set of coherent states, then there is an additional and alternative procedure to form the inner product between two elements, namely,

$$(\psi, \phi) = \int \psi(l)^* \phi(l) \, \delta l \ ,$$

for a suitable absolutely continuous measure δl on $\mathcal{L}$. In that case $\langle l | l' \rangle$ still plays the role of a reproducing kernel.

One can "reduce" a reproducing kernel to form a new function that can also serve as reproducing kernels for an associated Hilbert space composed of continuous functions. For example, one can fix certain elements of l, or perform weighted integrals over the same elements in l and l', etc. Such procedures lead to new, reduced reproducing kernels that can be used to generate associated reproducing kernel Hilbert spaces. Such techniques are very useful to pass from one Hilbert space to another Hilbert space, as we illustrate in the following Exercise.

Let

$$|p, q\rangle = e^{-iqP} \, e^{ipQ} \, |0\rangle$$

denote a set of coherent states, and focus on the overlap function

$$\langle p, q | p', q' \rangle \ .$$

a) Consider the three different reduced reproducing kernels given by

i) $\langle\!\langle q|q'\rangle\!\rangle \equiv \langle 0,q|0,q'\rangle$,

ii) $\langle\!\langle q|q'\rangle\!\rangle \equiv \langle 7,q|7,q'\rangle$,

iii) $\langle\!\langle q|q'\rangle\!\rangle \equiv \int \langle p,q|p',q'\rangle e^{-p^2-p'^2}\,dp\,dp'$,

and determine the dimensionality of the associated reproducing kernel Hilbert space in each case.

b) Some reproducing kernel Hilbert spaces admit alternative local integral representations to define the inner product – such is the case when the set $\{|l\rangle\}$ forms a set of coherent states. Show that the function

$$\langle q|q'\rangle = e^{-(q-q')^2}$$

defines a reproducing kernel that *cannot* have a local integral representation for the associated inner product.

Topic 2: Path Integrals

2.1 The free particle as well as the harmonic oscillator involve quadratic Lagrangians as well as quadratic Hamiltonians when expressed in traditional coordinates. As a consequence, a lattice version of path integrals associated with either of these problems involves the integral of a multi-dimensional Gaussian integral. Let $x = \{x_n\}_{n=1}^N$ denote an N-dimensional real-valued vector, and $\{G_{m,n}\}_{m,n=1}^{N,N}$ a symmetric, $N \times N$ positive-definite matrix. Consider the integral

$$I_N(s) := \int \exp[\,i\Sigma_m s_m x_m - \tfrac{1}{2}\Sigma_{m,n} x_m G_{m,n} x_n]\,\Pi_m dx_m\ .$$

Here $s := \{s_n\}_{n=1}^N$ denotes a fixed, real vector.

a) Evaluate the integral $I_N(s)$ as a function of the various variables.

Now consider the multi-dimensional integral

$$J_N(s) := \int \exp[\,i\Sigma_m s_m x_m + i\tfrac{1}{2}\Sigma_{m,n} x_m H_{m,n} x_n]\,\Pi_m dx_m\ ,$$

where we have added a factor of $-i$ before the quadratic term in the exponent, and relaxed the elements of the matrix $\{H_{m,n}\}$ such that it is symmetric, has a nonvanishing determinant, but no longer need be positive definite. Define this integral by adding a small additional damping term, e.g.,

$$H_{m,n} \to H_{m,n} + i\epsilon\delta_{m,n} \ , \quad 0 < \epsilon \ll 1 \ .$$

The proper definition of $J_N(s)$ is given by evaluation of the integral with a positive ϵ followed by a limit $\epsilon \to 0$ after the integral has been taken.

b) Evaluate the integral $J_N(s)$.

2.2 The result of the multi-dimensional integral dealt with in Exercise **2.1** consisted of an expression that logically divides into an amplitude factor and an exponent. The proper exponent can be obtained in a more convenient fashion as follows. As a Gaussian integral, the same exponent arises if one extremizes the exponent in the integrand.

a) Show that the resultant exponent is given by simply extremizing the exponent in the integrand.

The amplitude factor can also be determined in terms of a second-order partial derivative of the exponent determined in part a).

b) Find the appropriate second-order partial derivative expression of the exponent that determines the amplitude factor.

2.3 Use the results of Exercises **2.1** and **2.2** to evaluate:

a) The propagator for the harmonic oscillator of angular frequency ω as a path integral of the form

$$\langle x'', T | x', 0 \rangle = \mathcal{N} \int e^{i\frac{1}{2}\int_0^T [\dot{x}^2 - \omega^2 x^2]\, dt}\, \mathcal{D}x \ ,$$

where one integrates over "all paths" subject to the boundary conditions that $x(T) = x''$ and $x(0) = x'$. The resultant expression corresponds to the matrix element

$$\langle x'' | e^{-i\mathcal{H}T} | x' \rangle \ ,$$

where $\mathcal{H} = \frac{1}{2}[P^2 + \omega^2 Q^2]$.

The former path integral is a *configuration-space* path integral, which is to be contrasted with a *phase-space* path integral that we consider next.

b) Evaluate the propagator for the harmonic oscillator of angular frequency ω as a phase-space path integral of the form

$$\langle q'', T | q', 0 \rangle = \mathcal{M} \int e^{i \int_0^T [p\dot{q} - \frac{1}{2}(p^2 + \omega^2 q^2)]\, dt}\, \mathcal{D}p\mathcal{D}q \ ,$$

where one integrates over "all paths" such that $q(T) = q''$ and $q(0) = q'$, and in addition one integrates over *all* p-paths without any boundary conditions on those paths. Compare the results obtained in part b) with those obtained in part a).

c) Take the limit $\omega \to 0$ in either of the expressions in part a) or b), and thereby determine the propagator for the free particle of unit mass.

2.4 Let $x(t)$, $t \geq 0$, denote a *standard Brownian motion stochastic variable*, namely, a random-function variable that satisfies the following four properties:

i) $\langle x(t) \rangle = 0$

ii) $\langle x(t)x(s) \rangle = \min(t, s)$

iii) $x(0) = 0$

iv) $x(t)$ is a Gaussian random variable where the angle brackets $\langle \ \rangle$ denote ensemble average, i.e., an average over all Brownian motion paths.

a) Evaluate $\langle [x(t) - x(s)]^2 \rangle$

b) Using hypothesis iv) above, evaluate

$$\langle e^{i \int u(t) x(t)\, dt} \rangle$$

where $u(t)$ denotes a smooth function of time and the integral runs over $0 \leq t < \infty$.

The result of this stochastic average may also be described by an important Gaussian measure, the *Wiener measure* μ_W, implicitly defined by

$$\langle e^{i\int u(t)x(t)\,dt} \rangle := \int e^{i\int u(t)b(t)\,dt}\, d\mu_W(b) \; ,$$

where $b(t)$ denotes a functional realization of the paths that make up the stochastic variable $x(t)$. It follows that stochastic averages over random paths are equivalent to averages in the probability distribution afforded by the Wiener measure; stated otherwise, it follows that

$$\langle (\cdot) \rangle = \int (\cdot)\, d\mu_W$$

for any expression $(\cdot)$.

c) Based on the result of b), evaluate

$$\langle \exp\{-[x(t)-x(s)]^2/|t-s|^\beta\} \rangle \; .$$

d) Use the result of c) to show that

$$|x(t)-x(s)|^2 \le C\,|t-s|^{1+\epsilon} \; , \qquad 0 \le t, s \le 1 \; , \quad \epsilon > 0 \; ,$$

for some random variable C, with probability *one*, and that

$$|x(t)-x(s)|^2 \le C'\,|t-s|^{1-\epsilon} \; , \qquad 0 \le t, s \le 1 \; , \quad \epsilon > 0 \; ,$$

for some random variable C', with probability *zero*.

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The result of this exercise establishes that Brownian motion paths – or equivalently, a Weiner measure – is concentrated on paths that are continuous but for which the time derivative is undefined (i.e., infinite) for almost all time. The continuity and nondifferentiablity of Brownian motion paths are important and characteristic features of such paths.

Topic 3: Quantum Constraints

3.1 Consider the classical system described by the classical action

$$I = \int [p\dot{q} - \lambda p]\, dt \ ,$$

where p and q are the dynamical variables, the Hamiltonian vanishes, and the Lagrange multiplier λ enforces the single constraint $p = 0$.

a) Discuss the solution to this classical system and find which variables are gauge independent (do *not* depend on the choice of the Lagrange multiplier λ), and which variables are gauge dependent (*do* depend on the Lagrange multiplier λ).

In discussing the quantization of this system, let us first introduce canonical coherent states

$$|p, q\rangle = e^{-iqP}\, e^{ipQ}\, |0\rangle \ ,$$

where the fiducial vector is conveniently chosen as the solution to the equation $(Q + iP)|0\rangle = 0$. The overlap function of two such coherent states is given by

$$\begin{aligned}\langle p'', q''|p', q'\rangle &= \pi^{-1/2} \int_{-\infty}^{\infty} e^{-(k-p'')^2/2 + ik(q''-q') - (k-p')^2/2}\, dk \\ &= e^{i\frac{1}{2}(p''+p')(q''-q') - \frac{1}{4}[(p''-p')^2 + (q''-q')^2]} \ .\end{aligned}$$

We now describe the *projection operator method* to deal with quantum constraints. For a set of quantum constraints $\{\Phi_\alpha\}_{\alpha=1}^{A}$ the projection operator of interest is chosen as

$$\mathbb{E} = \mathbb{E}(\Sigma_{\alpha=1}^{A} \Phi_\alpha^2 \leq \delta(\hbar)^2) \ .$$

For the problem of present interest, we define the projection operator $\mathbb{E}$ as

$$\mathbb{E} = \mathbb{E}(P^2 \leq \delta^2) = \mathbb{E}(-\delta < P < \delta) \ .$$

The reduced reproducing kernel for this example is then

$$\begin{aligned}\langle p'', q''|\mathbb{E}|p', q'\rangle &= \pi^{-1/2}\int_{-\delta}^{\delta} e^{-(k-p'')^2/2+ik(q''-q')-(k-p')^2/2}\,dk \\ &= \frac{\sin((q''-q')\delta)}{\pi^{1/2}(q''-q')}\,e^{-(p''2+p'^2)/2} + O(\delta^2)\ ,\end{aligned}$$

valid for small δ, $0 < \delta \ll 1$. In order to avoid a vanishing limit as $\delta \to 0$, we rescale this expression and consider

$$\langle\!\langle p'', q''|p', q'\rangle\!\rangle \equiv \lim_{\delta\to 0}\frac{\langle p'', q''|\mathbb{E}|p', q'\rangle}{\langle 0|\mathbb{E}|0\rangle}\ ,$$

which leads to

$$\langle\!\langle p'', q''|p', q'\rangle\!\rangle = e^{-(p''2+p'^2)/2}\ .$$

The meaning of this expression is the following: As a product of a function of p'' and a function of p', this expression, when used as a reproducing kernel, leads to a *one*-dimensional physical Hilbert space.

Note well that the reduced reproducing kernel does *not* depend on either q'' or q'; this is a clear signal that, as far as the coherent state labels are concerned, in the physical Hilbert space we have reached a state where the coefficient of these missing variables is zero, i.e., "$P = 0$", as desired.

An observable operator $\mathcal{O}$ is one that commutes with the projection operator, i.e., $[\mathcal{O}, \mathbb{E}] = 0$. One can select the *observable part* of any operator $\mathcal{G}$, even one that may fail to commute with $\mathbb{E}$, simply by declaring the observable part of $\mathcal{G}$ to be $\mathcal{G}^E \equiv \mathbb{E}\mathcal{G}\mathbb{E}$.

b) For the problem at hand, determine the observable part of Q, namely, compute $Q^E = \mathbb{E}Q\mathbb{E}$.

If $\phi_1 = 0$ and $\phi_2 = 0$ denote two constraints, such constraints are called *irreducible* if they are not redundant. In other words, if $\phi_1 = 0$ does *not* imply $\phi_2 = 0$, or vice versa. On the other hand, a pair of constraints is called *reducible* if the vanishing of one of them already implies the vanishing of the other constraint. These notions may be generalized to any number of constraints.

c) As an example, consider the simple system with two constraints given by

$$I = \int [p\dot{q} - \lambda_1 p - \lambda_2 p]\, dt \ ,$$

which describes a classical system with two reducible – indeed *identical* – constraints. Describe (i) the classical theory for this system. Then find (ii) the reduced reproducing kernel that characterizes the physical Hilbert space. Compare the results for the classical and quantum theories for this case with the irreducible case described in the introduction to this problem.

d) As a modification of this example, consider the system described by

$$I = \int [p\dot{q} - \lambda_1 p - \lambda_2 p - \lambda_3 p]\, dt \ ,$$

which corresponds to a system with vanishing Hamiltonian and *three* identical constraints. Again, describe the classical and quantum theories and compare your results with those of the preceding example and the one in the introduction to this problem.

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The lesson of this example, is that the projection operator method easily handles reducible constraints and yields the same results as that obtained from an irreducible set of constraints.

3.2 In the classical theory, a constraint $\phi(p,q)$ is called *regular* whenever $\phi(p,q) = 0$, and also that both of the equations

$$\frac{\partial\phi}{\partial p} \neq 0 \ , \qquad \frac{\partial\phi}{\partial q} \neq 0$$

are everywhere valid on the constraint hypersurface. On the other hand, a constraint $\phi(p,q)$ is called *irregular* whenever $\phi(p,q) = 0$, but for which

$$\frac{\partial\phi}{\partial p} = 0 \ , \quad \text{and/or} \qquad \frac{\partial\phi}{\partial q} = 0$$

on the constraint hypersurface. Consider the following problem:

A classical system is defined by the action functional

$$I = \int [p\dot{q} - \lambda p^3] \, dt \; .$$

Again we have no Hamiltonian and a single – but *irregular* – constraint $p^3 = 0$.

a) Study the solution of the classical system. Find the gauge independent variables and the gauge dependent variables. Compare these results with those of Exercise **3.1** a.

b) Study the quantum theory of this system. In particular, introduce coherent states and a projection operator $\mathbb{E}$ appropriate to the present constraint. Evaluate the expression

$$\lim_{\delta \to 0} \frac{\langle p'', q'' | \mathbb{E} | p', q' \rangle}{\langle 0 | \mathbb{E} | 0 \rangle} \; ,$$

and compare the results with those of Exercise **3.1** b. Determine that the reduced reproducing kernel that arises for the present irregular constraint example generates the same physical Hilbert space as was found in the case of a regular version of the same constraint.

c) Repeat part a) and part b) above in the case that the classical system is characterized by the classical action

$$I = \int [p\dot{q} - \lambda p^\beta] \, dt \; ,$$

for any choice of real β such that $\beta > 1$, and which leads to an irregular constraint.

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The purpose of this exercise is to show that the projection operator method works just as easily for irregular constraints as it does for regular constraints.

3.3 Next let us combine reducible and irregular constraints. Consider the classical system described by the classical action

$$I = \int [p\dot{q} - \lambda_1 p^3 - \lambda_2 p^3]\, dt \; ,$$

which involves two identical irregular constraints leading again to a reducible system of constraints.

a) Discuss the classical solutions to this system. Identify the gauge independent variables and the gauge dependent variables.

b) Quantize the foregoing classical system. Determine a reproducing kernel for the physical Hilbert space. Compare your results with those of Exercises **3.1** and **3.2**.

* * *

This exercise is used to show that the projection operator method works well for reducible as well as irreducible constraints even when coupled with irregular or regular versions of those constraints.

3.4 It may seem artificial that constraints exist that are reducible. Consider the following classical system with *three* degrees of freedom, which we label by the canonical pairs (p, q), (r, s), and (u, v):

$$I = \int \{p\dot{q} + r\dot{s} + u\dot{v} - \lambda_1 p - \lambda_2[p + r(u^2 + v^2 - c)]\}\, dt \; .$$

When $u^2 + v^2 = k$, this system has two independent constraints $p = 0$ and $p + (k - c)r = 0$ when $k \neq c$, but effectively two identical constraints when $k = c$. In the quantum theory, the two constraints read

$$P \; , \qquad P + R(U^2 + V^2 - c) \; ,$$

In the sector where $U^2 + V^2 = 2m + 1$, $m = 0, 1, 2, \ldots$, it follows that the two constraints become

$$P \; , \qquad P + (2m + 1 - c)R \; .$$

and thus $k = 2m + 1$. If c is chosen as an odd positive integer, then we can arrange that $k = c$ for some choice of m. Thus, while the constraints are generally irreducible, there is indeed a nonzero subspace in which they act reducibly.

a) Discuss the classical system described above.

b) Discuss the quantization of this example, and derive a suitable reproducing kernel that characterizes the physical Hilbert space.

* * *

The several exercises in this section have focussed on examples that involve reducible and/or irregular constraints. It is noteworthy that the projection operator method is well suited to handle all these examples, and moreover, no other commonly used quantization procedure can readily handle reducible and irregular constraints involving both first and second class constraints. As the last example illustrates, it is quite possible that reducible constraints may enter in unexpected ways, and thus it is noteworthy that at least one quantization procedure is capable of successfully treating such systems.

In order to test other methods, proponents of alternative procedures of quantization of constrained systems are encouraged to try their method on the exercises given in this section.