

The Many Dimensions of Dimension

S. Deser

Department of Physics, Brandeis University,
Waltham, MA 02454, USA

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Abstract

Some aspects of the history and impact of the dimension revolution are briefly surveyed, starting with Nordström's (1914) $D=5$ scalar gravity-electromagnetism unification.

1 Introduction

Calling into question the very 4-dimensionality of spacetime was revolutionary and far-reaching: one of the great innovations of the 20th Century. That this was first done just nine decades ago (albeit only after much hesitation over its first fifty years) seems surprising, so quickly rooted has this concept become. In this lecture I propose to survey – necessarily very briefly and unsystematically – some of the stops on this evolving journey, as one might explain them to the pioneers come back to life. Consequently, I can hardly cover any of the more recent flood of related ideas, let alone their mounting phenomenological relevance. That belongs to a different lecture altogether. The story here is primarily devoted to the period ending with the early impact of strings and supergravity on the subject. But neither is this a historical lecture: priority and citation issues can be found elsewhere than in the present impressionistic excursion through our dimensional heritage.

Most conveniently, many of the older original papers can be found – and translated into English – in [1]. The first book covers Nordström's and many later works as well; the second is devoted to Oskar Klein's contributions. A very recent survey of higher D , especially in the context of particle physics phenomenology, is in [2]. Finally, [3] provides a current general analysis of consistency in dimensional reduction methods, together with a history.

2 History: Dimension from formal unification

It is particularly appropriate to begin this brief voyage through other dimensions with the pre-general relativity attempt, by Nordström [4], at formal unification of scalar gravity and electromagnetism. Although the gravity model was unrealistic, its key insight – dimension as unifier – has become a standard tool in all later attempts at finding “theories of everything”.

After 1905, it was clear that action-at-a-distance Newtonian gravity was inconsistent with special relativity. At the very least it had to be covariantized, schematically:

$$\nabla^2 \phi = \rho \quad \Rightarrow \quad \square \phi = T_{\mu}^{\mu} . \quad (1)$$

That the source of the wave equation is the trace of the matter stress tensor is pretty well forced, since it must be a 4-scalar that reduces to mass density in the nonrelativistic limit. The only other known force, electromagnetism, was represented by a four-vector potential A_μ . Nordström’s insight was to invent (1) and to unify¹ the two forces into a single 5-vector

$$(A_\mu, \phi) \rightarrow B_M, \quad M = (0, 1, 2, 3, 5), \quad (2)$$

in a D=5 space. As a result, the combined vector-scalar action of D=4 simply emerges from pure Maxwell in D=5, at least if one omits all x^5 dependence ($\partial/\partial x^5 = 0$) *ab initio*.² That is,

$$-\frac{1}{4} F_{MN}^2 \rightarrow -\frac{1}{4} F_{\mu\nu}^2 - \frac{1}{2} (\partial_\mu \phi \partial^\mu \phi) g^{55} \quad (3)$$

Note the 55-signature here; its spacelike (+) sign is absolutely essential, but must be put in by hand, to keeping (scalar) gravity ghost-free³ and – consequently – attractive! There is also an attendant (rather unnatural) source “unification”, with the 5-current j^M defined as (j^μ, T_α^α) . Allowing some – say periodic – dependence on x^5 in this precursor model, would have similar effects to those familiar from the Kaluza–Klein (KK) [5] unifications to which I now turn.

The Nordström lesson was rediscovered (independently) by K and K in the correct, Einstein gravity, context where the Maxwell vector potential was leapfrogged by the metric tensor:

$$(A_\mu, g_{\alpha\beta}) \rightarrow g_{MN} = (g_{\alpha\beta}, g_{\mu 5}, g_{55}). \quad (4)$$

Here dominance by the highest spin now costs us an extra, scalar, field beyond the desired tensor-vector doublet. At linearized approximation $g_{MN} = \eta_{MN} + h_{MN}$, the action decomposes into the sum of spins 2, 1 and 0 parts:

$$I_{KK} = \int d^5x \sqrt{-g} R(g_{MN}) \rightarrow I_{s=2}(h_{\alpha\beta}) + I_{s=1}(h_{\alpha 5}) + I_{s=0}(h_{55}). \quad (5)$$

All this is still in the $\partial/\partial x^5 = 0$ context and there remains some unnaturalness about the source T^{MN} , *e.g.*, how to identify T^{55} . Note that the correct choice of 55 signature simultaneously ensures non-ghost behavior for both $s=1$ and $s=0$.

Why did the pioneers stop at $D=5$? The answer is Occam’s razor: there were only two known forces, so not only did D=5 suffice but even it gave the (now superfluous) g_{55} field, which however prefigured later scalar-tensor gravity models. Indeed, as we shall see, even now we are not sure we know what is the right D , if that is even a correct question: the notion of dimension may dissolve below Planck scales.

3 Taking extra dimensions (really) seriously; D from groups?

As was especially emphasized by Klein’s innovations, one should not constrain higher D models to be x^5 -independent, or else the extension – however elegant – is just a formal one. Indeed, Klein was

¹Of course, A_μ was subject to local gauge changes $\partial_\mu \Lambda(x)$, and ϕ only to constant additions, but Nordström emphasized the field strengths ($F_{\mu\nu}$, $F_{\mu 5} = \partial_\mu \phi$).

²It was fortunately only discovered much later (see [3]) that “brutal” reductions to $D=4$ in which unwanted fields and “ x^5 ”-dependence are simply dropped is not always consistent!

³The long range common to both forces was Nordström’s driving element, even though the attraction-repulsion difference between them was not discussed. *That* little problem goes back – at least – to Maxwell, but the alternation of attraction/repulsion for even/odd spin intermediary fields was not properly understood until QFT times!

able to exploit x^5 -dependence in quite “modern” ways, especially his derivation of electric charge quantization as a result of assumed periodicity in x^5 . Furthermore, he was driven (albeit implicitly) to his now-famous “almost”-derivation (translated in [1]) of Yang–Mills (YM) theory by extension of these ideas. Ironically, it was only much later that a detailed “KK” inclusion (in suitably higher D) of YM was formulated – as a mere problem, in deWitt’s 1963 Les Houches lectures [6] (also in unpublished work of Pauli). This notion has been generalized to more abstract “internal” groups that can mimic real dimension through what can aptly be characterized as “dimensional transmutation”. Mixing of geometry and “internal” symmetries is clearly a fertile idea.

For us, dependence on ever higher dimensions, of whatever origins, has become the norm – we have much less respect for $D=4$ as the “true” arena underlying dynamics, especially after supergravity and superstrings (see below).

4 Compactify: a lot/a little?

We may not have much respect left for $D=4$, but we are still constrained to live and observe there. The direct presence of extra dimensions must be sweepable under some (macroscopic) rug. One way is making them very small and compactifying, as Klein originally did for x^5 . [This is the usual “garden hose” picture of extra dimension in popular science expositions.] More recently (see [2]) there have been explorations of models with relatively large – almost macroscopic – extra D , that surprisingly cannot be excluded using observed gravitational bounds. These remarks already point to the more general problem of too much freedom: what is *the* vacuum state and how is it fixed – why is compactification at all (let alone narrowing down to any specific one) the most attractive state? Perhaps this is another badly posed question – are there many vacua, all of which are of equal merit? In string theory, the plethora of Calabi-Yau spaces allowed within a given $D=10$ is currently a similar dilemma (see below). Indeed, we don’t even know in general why 4 big is “better” than say 3 big, or all D big! There is clearly a lot of unknown territory even at the “elementary” level.

5 The quantum input: dimension as anomaly-killer

Finding principles that actually fix the dimensionality of space-time (and perhaps even its signature) is an old and as yet unachieved goal. However, one mechanism stands out in this quest, because it is simultaneously (1) a clear application of quantum field theory requirements, (2) leads to a unique dimensionality, and (3) forms the basis of superstring theory. The trick lies in first transmuting D into an index that counts the number of excitations – here scalar and spinor – living in a lower – here $D=2$ – spacetime. As we all know, a (bosonic) string is a set of one-dimensional objects evolving in time; very roughly,

$$I_s \sim \int \int d\sigma d\tau \partial_\mu \phi^a \partial_\nu \phi^b g_{ab} g^{\mu\nu} \sqrt{g_2}, \quad a, b = 1 \dots D. \quad (6)$$

The index a represents the ϕ fields as the coordinates in a “target space” with metric g_{ab} and of dimension D , where we live, while μ ranges over the 2 dimensions (τ, σ) of the world sheet on which each ϕ^a and the intrinsic 2-metric depend. Now in QFT, classical symmetries, such as the conformal invariance of a scalar field in two dimensions, best exhibited by writing it in a gravitational background,

$$I = -\frac{1}{2} \int d^2x \sqrt{-g} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi, \quad \delta I = 0, \quad g_{\mu\nu} \rightarrow S(x) g_{\mu\nu}, \quad (7)$$

can be, and usually are, broken (hence “anomalies”) by the necessity of introducing a regularization scale or cutoff to make sense of the closed loops. This generation of anomalies is a very deep, nonperturbative, phenomenon that has direct – and observed – physical consequences. A consistent quantum theory must be anomaly-free; for the above bosonic string model, the coefficients of the conformal anomaly add up to $(D-26)$; bosonic strings have other problems (such as tachyonic modes) however, that are absent in their supersymmetric incarnations. These latter scalar (ϕ^a) plus spinor (ψ^a), systems yield coefficients that add up to $(D-10)$. Since we live in the target spaces, ours is a $D=10$ world. Even if this number is slightly off our observed $D=4$, it is one of superstring theory’s impressive predictions: a unique and not outlandish value of D , in addition to its other wonderful (we hope!) attributes.⁴

6 Embedding/Braneworlds

Dealing with this enormous highly current topic can only be too brief or too long, so I opt for the former (and refer to [2] for more): once we are committed to a truly $D > 4$ world, then our observed $D=4$ “flatland” can have links to itself through those other, $D-4$, dimensions: we live in a submanifold rather than in a simple direct product, $4 \otimes (D - 4)$, space. Modern exploitations of this go under the rubric of braneworlds, subspaces with considerably richer structure than $D=4$ alone would allow. For example, we would observe the usual power laws $\sim 1/r$ for gravitational or electric potentials, while significant deviations from them could be present “nearby”. Normally, in D -spacetime, the Coulomb potential is $r^{-(D-3)}$, yet there are various ways, besides a simple direct product, to feel just $1/r$ in our particular corner.

7 “Dynamical” Compactification

The big questions we have already encountered in our survey is how and why there are (at least at the present time) precisely $D=4$ macroscopic dimensions where our laws apply, and if one is greedy, why our signature is $(- + + +)$. We have also noted that these questions are still not only open, but possibly ill-posed! Here is one possible way to distinguish $D=4$. Consider the popular value of D , the context of supergravity (SUGRA), namely $D=11$, which will be separately treated below. This value is the only competitor to $D=10$; indeed one of the current active areas is the relation of the string $D=10$ and the SUGRA $D=11$, evoking the (mysterious) M -theory I will not mention further. Although 11 is not *a priori* more amenable than 10 to compactification notions, there is one possible mechanism here [7], that illustrates possible attacks on the problem. We want to understand why there is a 4+7 breakup, rather than say 7+4 (or any other) with 4 big(ger) dimensions. Now one component of $D=11$ SUGRA is a 3-form potential A_{MNP} and its attendant 4-form field strength $B_{MNPQ} = \partial_{[Q}A_{MNP]}$. A caricature of the idea is that some sort of symmetric vacuum breaking is caused by this totally antisymmetric B -form, which thereby separates $D=4$ from the rest, *i.e.*, $\langle B^{MNPQ} \rangle \sim \epsilon^{\mu\nu\alpha\beta}$. That is, the content of the system has built-in symmetry breaking potential (just as Higgs actions do, in another story). This suggestion has the merit of showing how dynamics alone could dictate its own “optimal” D ; perhaps the breakup is even epoch-dependent and we just happen to live in the 4+7 phase.

⁴String theory is even better served by anomaly-freedom than this: dangers from different would-be anomalies reduce the number of possible strings to just five, which in turn are linked by dualities into an essentially unique model.

8 SUGRA: Upper Bounds on D

Having gone from $D=4$ to ever higher dimensions, one begins to wonder why there should be *any* upper bound to D at all (we will mention the limit $D \rightarrow \infty$ below), except in the superstring case, where we encountered a precise value, $D=10$, rather than merely a bound. We now consider the very important related area – supersymmetry (SUSY) and SUGRA – in which a whole spectrum of dimensions is possible, but in which an upper bound is in fact set by physical considerations of a down-to-earth kind. We recall that SUSY is the study of Bose–Fermi matter systems, including spins $(0, \frac{1}{2}, 1)$ while SUGRA deals with spin $3/2$ and its spin 2 gravitational partner, possibly including lower spins as well. In SUSY the symmetry is under constant (fermionic) transformations, while SUGRA adds to general coordinate invariance a sort of “Dirac square-root” local fermionic parameter. One of the key requirements for a system to obey a supersymmetry algebra is one-to-one matching of bosonic and fermionic degrees of freedom (DoF). This is what limits D , simply because the number of tensors and of spinors grow very differently with dimension. Roughly, the former grow as a power, $\sim D^2$, the latter exponentially $\sim 2^D$. Consider first the original [8], and simplest (“ $N=1$ ”) SUGRA, in $D=4$. Here the system is the sum of Einstein⁵ and massless spin $3/2$ fields. The graviton and fermion are separated by a $1/2$ unit of spin and rotate into each other; schematically the transformation

$$\delta g_{\mu\nu} = \alpha(x)(\gamma_\mu \psi_\nu + \gamma_\nu \psi_\mu), \quad \delta \psi_\mu = D_\mu \alpha(x),$$

where $\alpha(x)$ is a fermionic parameter, leaves the action invariant. In $D=4$ all massless particles have helicities $\pm s$ only, *i.e.*, they all (except scalars) have 2 DoF, so the numbers match. However, as D grows, the number of Einstein graviton DoF goes as $\frac{D(D-3)}{2}$, the number of transverse-traceless components of the spatial $(D-1)$ metric. Instead, ψ_μ is a (vector-)spinor and shares the $\sim 2^D$ (times the vector) growth of spinors. The only way to keep bose/fermi DoF parity is to introduce more lower spin fields as part of the supergravity multiplet, something that can also be done at $D=4$, but as an option rather than necessity. One can tabulate [10] all fields’ DoF at any D , so it is easy to count the options in general. From this point of view, $D=11$ is [11] the highest dimension where the balance is kept without making appeal either to spin > 2 or to more than one graviton or both. Now from the dynamical point of view, both of those latter choices are forbidden: the coupling of gauge fields with $s > 2$ to gravity is inconsistent, and more than one gravity, *i.e.*, a sum of two separate Einstein metric actions is – as might be expected – not physical. Hence it is simply “ordinary” dynamics, rather than the impossibility of a kinematical spin matching in $D > 11$ that forbids SUGRA there, a very appealing outcome. Also, at $D=11$, there is but one way to balance the 44 gravitons in the bosonic sector to the 128 ψ_μ components, namely to add 84 3-form fields $A_{\mu\nu\alpha}$. This uniqueness is one the model’s lasting attractions.⁶

9 $D \rightarrow \infty$?

This is our shortest section. One cannot avoid wondering what sort of world would exist for $D \rightarrow \infty$. There are only two references [13] to my knowledge, at least in the context of Einstein gravity, with

⁵There is also a version incorporating a necessarily negative (Anti-deSitter) cosmological term along with an (apparent) fermion mass term [9].

⁶So unique is this theory, in fact, that it is the one SUGRA, and indeed the one known physical model, that does not even permit a cosmological extension at all [12]. It also forbids supermatter sources.

no very specific indication or conclusion as yet; this may be a sign from heaven, to stick to finite D , or perhaps we don't yet know how to extract the right questions here either.

10 D and Anthropic Principles

The trend of our survey so far has been that there is no really compelling theory of D (except perhaps that from strings), nor of the $D=4+(D-4)$ breakup for a given D , nor of the signature of spacetime. It is therefore worth mentioning a currently fashionable idea, loosely styled anthropic: Let a thousand vacua bloom; in the ensemble of all these possible cosmologies, we're in the (only?) one that allows us. In this view, there is not point seeking some single string vacuum among the immense multitude present in various Calabi–Yau spaces with various, fluxes, brane choices, etc., provided that one of them is ours! This statistical surrender to the counsel of despair may yet be necessary, but it should perhaps not (yet) be welcomed. The most recent, if hardly final, salvo in this battle is [14].

11 $D < 4$: Laboratories of Real Physics

We have, so far, explored $D > 4$ worlds; what about dimensionally challenged spacetimes with $D < 4$? Even $D=1$, while cramped, is big enough for ordinary quantum mechanics: $I = \frac{1}{2} \int dt \dot{\mathbf{r}}^2(t)$ describes a $D=1$ base manifold with a $D=3$ target space (or $D=4$ spacetime), while we have seen that $D=2$ contains nothing less than string theory. This leaves $D=3$, a dimension that contains real condensed matter physics, as described by planar QED, enhanced by an extremely relevant novel term, the Chern–Simons (CS) invariant,⁷

$$I_{CS} = \int d^3x \epsilon^{\mu\nu\alpha} A_\mu \partial_\nu A_\alpha = \int d^3x A_\mu {}^*F^\mu \quad (8)$$

where ${}^*F^\mu$ is the (vector) dual of $F_{\nu\alpha}$. Its nonabelian counterpart is even more topologically endowed; they are of particular interest also in the finite temperature context, a rich and mathematically complex area.

While $D=3$ Einstein gravity (and SUGRA) are devoid of dynamics as they stand, because Ricci and Riemann tensors are equivalent in $D=3$, yet here too (gravitational) CS terms exist and provide true DoF when added to the Einstein action. In both vector and tensor settings cases, novel effects such as coexistence of non-zero mass with gauge invariance, abound [15]. So $D=3$ can, in some ways, be regarded as the residue of a “KK” $D=4$ manifold, with the CS terms descending from topological invariants like $\int d^4x F_{\mu\nu} {}^*F^{\mu\nu}$, but that is a whole other story. One lesson for us, in any case, is the D -dependence of interesting possible invariants: Not only do conventional actions exist in various dimensions, but some do so in only one particular D , others (like Gauss–Bonnet invariants) do so only above a given D .

⁷Historically, the first appearance of a CS term was in the SUGRA context, via the 3-form of $D=11$. There, however it is trilinear, rather than quadratic, in the fields.

Conclusion

In the preceding episodes, I have tried to sample, in a very nonsystematic way, some of the legacy and promise of the great concept that the D of physical theories need not be the same as that of the strictly $D=4$ world we seem to inhabit. This generalization fits the lesson of effective actions in physics: the more encompassing a new level of physical law, the less its concepts need resemble those of its limiting approximations. Indeed, one could argue that the freeing of dimensionality may be one of the most permanent new “top-down” ideas created in the past century: We are deeply indebted to its pioneers, Nordström, Kaluza and Klein.

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